WKB Approximation (Semiclassical)

identify regions in which wavelength is much shorter than typical distance over which potential energy varies

1D: $$\frac{d^2 u_E}{dx^2} + \frac{2m}{\hbar^2}\left(E - V(x)\right) u_E(x) = 0$$

define: $$k(x) \equiv \sqrt{\frac{2m}{\hbar^2}\left(E - V(x)\right)} \qquad E > V(x) \qquad \text{"allowed region"}$$ II

$$k(x) = -i\sqrt{\frac{2m}{\hbar^2}\left(V(x) - E\right)} \qquad E < V(x) \qquad \text{"forbidden region"}$$ I + III

$$= -i\kappa(x)$$

$$\rightarrow \frac{d^2 u_E}{dx^2} + [k(x)]^2 u_E(x) = 0$$

if $V(x)$ does not change with x (is constant) $\rightarrow$ $k(x)$ constant

$\rightarrow u_E(x) \propto e^{\pm ikx}$

if $V(x)$ varies slowly with x $\rightarrow$ try solution of form

$$u_E(x) = e^{\frac{i}{\hbar}W(x)}$$

$$\rightarrow i\hbar \frac{d^2W}{dx^2} - \left(\frac{dW}{dx}\right)^2 + \hbar^2[k(x)]^2 = 0$$ SE-like equation

Make approximation (explanation later)

$$\hbar \left|\frac{d^2W}{dx^2}\right| \ll \left|\frac{dW}{dx}\right|^2$$

$\rightarrow$ lowest order approximation $W_0'(x) = \pm \hbar k(x)$

$\rightarrow$ 1st order approximation: $$\left(\frac{dW_1}{dx}\right)^2 = \hbar^2[k(x)]^2 + i\hbar W_0''(x)$$
$$= \hbar^2[k(x)]^2 \pm i\hbar^2 k'(x)$$

term #1 >> term #2

$$\rightarrow W(x) \approx W_1(x) = \pm \hbar \int^x dx' \left[k^2(x') \pm i k'(x')\right]^{\frac{1}{2}}$$

$\sqrt{1-\varepsilon} = 1 - \frac{1}{2}\varepsilon$

$$\approx \pm \hbar \int^x dx'\, k(x') \left[1 \pm \frac{i}{2} \frac{k'(x')}{k^2(x')}\right]$$

$$= \pm \hbar \int^x dx'\, k(x') + \frac{i}{2} \hbar \ln\left[k(x)\right]$$

→ wavefunction in WKB limit to first order

$$u_E(x) \approx e^{\frac{i}{\hbar} W(x)} = \frac{1}{\sqrt{k(x)}} \exp\left[\pm i \int^x dx'\, k(x')\right]$$

→ specifies two solutions (±)

1. for region $E > V(x)$
2. for region $E < V(x)$

→ we need to join them: at classical turning point $\lambda = \frac{1}{k} \rightarrow \infty$ potential cannot be thought of as slowly varying

→ approximate region around $E \approx V$ with linear potential

[→ SE in appropriately scaled units: $\frac{d^2 u_E}{dz^2} - z\, u_E(z) = 0$

→ solutions are Airy functions]

Asymptotic dependences of Airy functions:

$$Ai(z) \rightarrow \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \exp\left[-\frac{2}{3} z^{\frac{3}{2}}\right] \qquad z \rightarrow \infty$$

$$Ai(z) \rightarrow \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} \cos\left(\frac{2}{3}|z|^{\frac{3}{2}} - \frac{\pi}{4}\right) \qquad z \rightarrow -\infty$$

$$\mathrm{I} \rightarrow \mathrm{II} \quad \frac{1}{(V(x)-E)^{\frac{1}{4}}} \exp\left[-\frac{1}{\hbar}\int_x^{x_1} dx' \sqrt{2m\left[V(x')-E\right]}\right] \rightarrow \frac{2}{(E-V(x))^{\frac{1}{4}}} \cos\left[\frac{1}{\hbar}\int_{x_1}^{x} dx' \sqrt{2m(E-V(x'))} - \frac{\pi}{4}\right]$$

$$\mathrm{III} \rightarrow \mathrm{II} \quad \left[\frac{1}{(V(x)-E)^{\frac{1}{4}}}\right] \exp\left[-\frac{1}{\hbar}\int_{x_2}^{x} dx' \sqrt{2m\left[V(x')-E\right]}\right] \rightarrow \left[\frac{2}{(E-V(x))^{\frac{1}{4}}}\right] \cos\left[-\frac{1}{\hbar}\int_x^{x_2} dx' \sqrt{2m(E-V(x'))} + \frac{\pi}{4}\right]$$

matching two solutions across turning point very difficult

1. Make linear approximation of potential around turning point
2. find solutions to SE in this region
3. match solutions to other solutions choosing appropriate integration constants.

case $I \to II$ ...

$III \to II$ ...

→ note that form in region II must be identical

→ argument differ by integer multiple of π ([illegible] ; [illegible]) both good

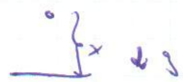
→ consistency condition

$$\int_{x_1}^{x_2} dx \sqrt{2m[E-V(x)]} = (n+\tfrac{1}{2})\pi\hbar \qquad (*)$$

→ fixes energy spectrum

Example: bouncing ball (bouncing neutron)

$$V = \begin{cases} mgx & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$



1^{st} idea: use $(*)$ with $x_1 = 0$, $x_2 = \frac{E}{mg}$

→ but does not work since no leaking into surface!

2^{nd} idea: find odd-parity solutions for $V(x) = mg|x|$

→ turning points $x_1 = -\frac{E}{mg}$, $x_2 = \frac{E}{mg}$

$$\to \int_{-E/mg}^{E/mg} dx \sqrt{2m(E - mg|x|)} = (n_{odd} + \tfrac{1}{2})\pi\hbar \quad (n_{odd} = 1, 3, 5, \dots)$$

or equivalently

$$\int_0^{E/mg} dx \sqrt{2m(E - mgx)} = (n - \tfrac{1}{4})\pi\hbar \qquad n = 1,2,3,4,\dots$$

$$\to \quad E_n = \frac{3(n-\frac{1}{4})^{\frac{2}{3}}}{2}(mg^2\hbar^2)^{\frac{1}{3}}$$ quantized energy levels of bouncing ball

compare to exact numerical solutions:

n	WKB	Exact
1	2.320	2.338
2	4.082	4.088
⋮		
10	12.828	12.827

← almost perfect

Example: quarkonium
bound state of quark - antiquark
$mg \Leftrightarrow$ free Schwinger...
$1.6 \cdot 10^5$ N (16 tons)
100 gr ball: 0.98 N

Outstanding: interpretation of $\hbar\left|\frac{d^2W}{dx^2}\right| \ll \left|\frac{dW}{dx}\right|^2$

becomes exact in limit $\hbar \to 0$ (classical limit)

time dependent wf:

$$\psi(x,t) \propto u_E(x)e^{-\frac{i}{\hbar}Et} = e^{\frac{i}{\hbar}W(x) - \frac{i}{\hbar}Et}$$

in Hamilton Jacobi formalism: $W \mathrel{\hat=}$ Hamilton's characteristic function

→ condition describes classical limit

also: condition the same as $|k'(x)| \ll |k^2(x)|$ (see lowest order WKB)

$$\frac{\lambda}{2\pi} = \frac{\hbar}{\sqrt{2m\{E-V(x)\}}} \ll \frac{2\{E-V(x)\}}{dV/dx}$$

de Broglie wavelength must be small compared to characteristic distance over which potential varies