(1) A solid ball starts from rest at the upper end of the track shown and rolls without slipping until it rolls off the right-hand end. If $H = 6.0 \text{ m}$ and $h = 2.0 \text{ m}$, what is its speed when it rolls off the end?

The total mechanical energy of the ball will be conserved – we must just make sure to include both its rotational and translational kinetic energies, and also the relationship between the speed of the centre of mass and the angular speed of the rotation for rolling without slipping, $v = R\omega$:

\[
\begin{align*}
\text{KE}_i + \text{PE}_i &= \text{KE}_i + \text{PE}_i \\
0 + mgH &= \frac{1}{2}mv_f^2 + \frac{1}{2}I_\omega^2 + mgh \\
mgH &= \frac{1}{2}mv_f^2 + \frac{1}{2}\left[\frac{2}{5}mR^2\right]\left[\frac{v_f}{R}\right]^2 + mgh \\
g(H - h) &= \frac{1}{2}v_f^2 + \frac{1}{5}v_f^2 = \frac{7}{10}v_f^2 \\
v_f &= \sqrt{\frac{10}{7}g(H - h)} \\
&= \sqrt{\frac{10}{7}(9.8 \text{ m/s}^2)(4.0 \text{ m})} \\
&= 7.48 \text{ m/s}
\end{align*}
\]

(2) A hollow sphere of radius 10 cm and mass 20 kg is mounted so that it can rotate freely about a horizontal axis that is parallel to and 5.0 cm from the central axis of the sphere. (a) What is the moment of inertia of the sphere about this axis? (b) If the sphere is released from rest with its central axis at the same height as the rotation axis, what is the angular speed of the sphere as it passes through its lowest position?

(a) We can use the parallel-axis theorem, which relates the moment of inertia we are looking for, $I$, with the moment of inertia for rotation about an axis parallel to ours and going through the centre of mass, $I_{com}$ ($h$ is the distance between the two axes):

\[
I = I_{com} + Mh^2
\]
\[
= \frac{2}{3}MR^2 + M\left(\frac{1}{2}R\right)^2 = \frac{2}{3}MR^2 + \frac{1}{4}MR^2
\]
\[
= \frac{11}{12}MR^2 = \frac{11}{12}(20 \text{ kg})(0.10 \text{ m})^2
\]
\[
I = 0.183 \text{ kg m}^2
\]

(b) Now use conservation of energy; the height of the centre of mass when the sphere passes through its lowest position is 5 cm lower than its initial height:
\[
\begin{align*}
\text{PE}_i + \text{KE}_i &= \text{PE}_f + \text{KE}_f \\
Mgh_i + 0 &= Mgh_f + \frac{1}{2}I\omega^2 \\
\frac{1}{2}I\omega^2 &= Mg(h_i - h_f) \\
\frac{1}{2}\left[\frac{11}{12}MR^2\right]\omega^2 &= Mg(h_i - h_f) \\
\omega^2 &= \frac{2g(h_i - h_f)}{11R^2} \\
\omega &= \sqrt{\frac{2g(h_i - h_f)}{11R^2}} \\
&= \sqrt{\frac{24(9.8 \text{ m/s}^2)(0.050 \text{ m})}{11(0.10 \text{ m})^2}} \\
\omega &= 10.3 \text{ rad/s}
\end{align*}
\]

(3) A hoop rolls down a ramp. (a) What must be the inclination of the ramp in order for the translational acceleration of the center of the hoop to have a magnitude of \(0.10g\)? (b) If a frictionless block were to slide down the incline tilted at that same angle, would its acceleration be more than, less than or equal to \(0.10g\)? Explain why without actually calculating the acceleration of such a block.

(a) We derived in class an expression for the translational acceleration of an object rolling down an incline (it is also given on p. 250 of Halliday, Resnick & Walker):

\[
a_{com} = \frac{g\sin\theta}{1 + I_{com}/MR^2}
\]

Here \(a_{com}\) is the acceleration of the centre of mass down the ramp and \(I_{com}\) is the moment of inertia of the rolling object for rotation about an axis passing through its centre of mass. In the case of a hoop, this is \(I_{com} = MR^2\). We can now find the needed inclination of the ramp:

\[
\begin{align*}
a_{com} &= \frac{g\sin\theta}{1 + MR^2/10R^2} \\
a_{com} &= \frac{1}{2}g\sin\theta \\
\sin\theta &= \frac{2a_{com}}{g} \\
\sin\theta &= \frac{2(0.10g)}{g} = 0.20 \\
\theta &= 11.5^\circ
\end{align*}
\]

(b) The acceleration of the block would be greater than that of the hoop. The reason is that the acceleration comes about due to a transformation of the initial gravitational potential energy of the object going down the ramp into kinetic energy. In the case of both the block and the hoop, the same amount of gravitational potential energy is transformed into kinetic energy – in the case of the block, all of the kinetic energy that is gained is translational, but in the case of the hoop, some of the kinetic energy that is gained is rotational, which means that the hoop has less translational kinetic energy after going the same distance down the ramp.
(4) A 3.0 kg object with velocity \( \mathbf{v} = (5.0 \text{ m/s})\mathbf{i} - (6.0 \text{ m/s})\mathbf{j} \) is located at \( x = 3.0 \text{ m}, y = 8.0 \text{ m} \) and experiences a force \( F = 7.0 \text{ N} \) acting in the negative \( x \) direction. (a) What is the angular momentum of the object about the origin? (b) What torque about the origin acts on the particle due to \( \mathbf{F} \)? (c) At what rate is the angular momentum of the particle changing with time?

(a) The diagram shows the object, its velocity and the force acting on it. The object’s momentum is in the same direction as its velocity, and is equal to \( \mathbf{p} = m\mathbf{v} \). The angular momentum of the object about the origin will be (\( \phi \) is the angle between \( \mathbf{r} \) and \( \mathbf{p} \))

\[
\mathbf{L} = \mathbf{r} \times \mathbf{p} \\
L = r p \sin \phi = r p \sin(\alpha + \beta) \\
r = \sqrt{(3.0 \text{ m})^2 + (8.0 \text{ m})^2} = 8.54 \text{ m} \\
v = \sqrt{(5.0 \text{ m/s})^2 + (-6.0 \text{ m/s})^2} = 7.81 \text{ m/s} \\
L = (8.54 \text{ m})[(3.0 \text{ kg})(7.81 \text{ m/s})] \sin(119) \\
L = 175 \text{ kg m}^2/\text{s}
\]

The direction of the angular momentum is given by the right-hand rule, and is into the page.

(b) The torque exerted by the force \( \mathbf{F} \) is

\[
\mathbf{\tau} = \mathbf{r} \times \mathbf{F} \\
\tau = r F \sin \phi = r F \sin(180 - \alpha) \\
\tau = (8.54 \text{ m})(7.0 \text{ N})(\sin(111)) \\
\tau = 55.8 \text{ N m}
\]

The direction of the torque is given by the right-hand rule, and is out of the page.

(c) The rate at which the angular momentum of the particle is changing with time is given by

\[
\frac{dL}{dt} = \tau = 55.8 \text{ N m}
\]

(5) A sanding disk with moment of inertia \( I = 1.2 \times 10^{-3} \text{ kg m}^2 \) is attached to an electric drill whose motor exerts a constant torque of \( \tau = 16 \text{ N m} \). What are (a) the angular momentum of the disk about its central axis 33 ms after the motor is turned on and (b) the angular speed of the disk at this same time?

(a) The angular momentum of the disk will be given by the expressions \( \tau = \Delta L/\Delta t \) or \( L = I \omega \); the first of these is easier to use based on the information given:

\[
\Delta L = \tau \Delta t \\
L - 0 = (16 \text{ N m})(33 \times 10^{-3} \text{ s}) \\
L = 0.53 \text{ N m s} = 0.53 \text{ J s}
\]
(b) Now that we have $L$, we can use the second of the relations given above to find $\omega$ – the angular speed of the disk at the motor has been going 33 ms:

$$\omega = \frac{L}{I}$$

$$= \frac{0.53 \text{ J s}}{1.2 \times 10^{-3} \text{ kg m}^2}$$

$$= 4.4 \times 10^2 \text{ rad/s}$$

Another way to get this is from the relation $\tau = I\alpha$:

$$\alpha = \frac{\tau}{I}$$

$$= \frac{16 \text{ N m}}{1.2 \times 10^{-3} \text{ kg m}^2}$$

$$= 1.33 \times 10^4 \text{ rad/s}^2$$

$$\omega = \omega + \alpha \Delta t$$

$$= 0 + (1.33 \times 10^4 \text{ rad/s}^2)(33 \times 10^{-3} \text{ s})$$

$$= 4.4 \times 10^2 \text{ rad/s}$$

(6) A small puck of mass $m = 0.200 \text{ kg}$ slides in a circle on a frictionless table while attached to a hanging mass $M_1 = 0.500 \text{ kg}$ by a cord through a hole in the table, as shown. (a) With what speed must the puck move to keep the hanging mass stationary if the radius of the circle in which it moves is $r_1 = 0.500 \text{ m}$? (b) What is the puck’s angular momentum in this case? Another mass $M_2$ is gently hung below $M_1$, pulling on the cord and causing the radius of the circular motion of the puck to decrease to $r_2 = 0.400 \text{ m}$. (c) What is the new velocity of the puck? (d) What is $M_2$?

(a) The upward tension in the cord must balance the downward force of gravity. This tension will also be equal to the centripetal force:

$$T - M_1 g = 0$$

$$\frac{mv^2}{r} - M_1 g = 0$$

$$v = \sqrt{\frac{M_1 rg}{m}} = \sqrt{\frac{(0.500 \text{ kg})(0.500 \text{ m})(9.8 \text{ m/s}^2)}{0.200 \text{ kg}}}$$

$$v = 3.50 \text{ m/s}$$

(b) Treating the puck as a point mass moving in a circle, its angular momentum will be

$$L = mvr$$

$$L = (0.200 \text{ kg})(3.50 \text{ m/s})(0.500 \text{ m}) = 0.350 \text{ kg m}^2/\text{s}$$

(c) The angular momentum of the system is conserved. Therefore,
\[ L_i = L_f \]
\[ mvr_i = mvf r_f \]
\[ v_f = \frac{v_i r_i}{r_f} \]
\[ v_f = (3.50 \text{ m/s}) \frac{0.500}{0.400} = 4.38 \text{ m/s} \]

(d) We can now find the mass \( M_2 \) using the same approach as in part (a)

\[
T - (M_1 + M_2)g = 0
\]
\[
\frac{mv^2}{r} - (M_1 + M_2)g = 0
\]
\[
M_2 = \frac{mv^2}{rg} - M_1
\]
\[
M_2 = \frac{(0.200 \text{ kg})(4.38 \text{ m/s})^2}{(0.400 \text{ m})(9.8 \text{ m/s}^2)} - 0.500 \text{ kg}
\]
\[
M_2 = 0.479 \text{ kg}
\]

(7) A wheel with a moment of inertia \( I \) is rotating freely at an angular speed of 800 rev/min on an axle with negligible moment of inertia. A second wheel with moment of inertia \( 2I \) that is initially at rest is then suddenly coupled to the same axle, so that it also rotates. (a) What is the angular speed of the combination of the two wheels rotating about the axle? (b) What fraction of the original rotational kinetic energy is lost?

(a) The key is that the angular momentum of the two wheels is conserved when they are coupled:

\[
L_{\text{before}} = L_{\text{after}}
\]
\[
I_{1i} \omega_{1i} + I_{2i} \omega_{2i} = I_{1f} \omega_{1f} + I_{2f} \omega_{2f}
\]
\[
I \omega_{1i} + 2I(0) = (I + 2I) \omega_f
\]
\[
\omega_f = \frac{\omega_{1i}}{3} = 267 \text{ rev/min}
\]

(b) The kinetic energy that is lost is given by

\[
\Delta KE = KE_{\text{before}} - KE_{\text{after}}
\]
\[
= \left( \frac{1}{2} I_{1i} \omega_{1i}^2 + \frac{1}{2} I_{2i} (0)^2 \right) - \frac{1}{2} I_f \omega_f^2
\]
\[
= \frac{1}{2} I \omega_{1i}^2 - \frac{1}{2} (3I) \omega_f^2
\]

and the fraction that is lost is given by

\[
\frac{\Delta KE}{KE_i} = \frac{\frac{1}{2} I \omega_{1i}^2 - \frac{1}{2} (3I) \omega_f^2}{\frac{1}{2} I \omega_{1i}^2}
\]
\[
= \frac{\omega_{1i}^2 - 3\omega_f^2}{\omega_{1i}^2}
\]
\[
= \frac{(800 \text{ rev/min})^2 - 3(267 \text{ rev/min})^2}{(800 \text{ rev/min})^2}
\]
\[
= 0.66
\]

66% of the initial kinetic energy is lost when the wheels are coupled.
(8) A board is balanced as is shown above. $m_1$ and $m_2$ are $x_1 = 2.5$ m and $x_2 = 1.2$ m to the left of the support point, which is just under the centre of the board. How far is $M$ from the support point?

We have three conditions for the equilibrium:

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum \tau = 0$$

In this case, there are no horizontal forces, so we only have the last two of these equations.

(i) $\sum F_y = 0$

$$F_{\text{support}} + m_1 g + m_2 g + M g + m_{\text{board}} g = 0$$

where $F_{\text{support}}$ is the upward force of the support and $m_{\text{board}}$ is the mass of the board (which we do not know).

(ii) $\sum \tau = 0$

We can choose any point about which to take the torques. It is convenient in this case to choose the centre of the board as the point about which we take our torques. The reason is that if we choose any other point, we will get a torque equation that has three unknowns: $F_{\text{support}}$, $m_{\text{board}}$ and $X$, where $X$ is the distance from the support to the mass $M$. This means that we’ll have two equations with three unknowns, since we are not given the mass of the board. If we take torques about the centre of the board, neither the weight of the board nor $F_{\text{support}}$ exert torques, and the only unknown in our torque equation will be $X$:

$$\sum \tau = m_1 g x_1 \sin(90) + m_2 g x_2 \sin(90) - M g X \sin(90) = 0$$

$$X = \frac{m_1 x_1 + m_2 x_2}{M} = 0$$

$$X = \frac{(1.5 \text{ kg})(2.5 \text{ m}) + (2.0 \text{ kg})(1.2 \text{ m})}{4.0 \text{ kg}} = 1.54 \text{ m}$$

(1b) A bowler throws a bowling ball of radius $R = 11$ cm along a lane. The ball slides on the lane with an initial speed $v_0 = 8.5$ m/s and initial angular speed $\omega_0 = 0$. The coefficient of kinetic friction between the ball and lane is $\mu_k = 0.21$. The kinetic friction force acting on the ball causes a linear acceleration of the ball while also producing a torque that causes an angular acceleration of the ball. When the speed $v$ has decreased enough and the angular speed $\omega$ has increased enough, the ball stops sliding and then rolls smoothly. (a) What then is $v$ in terms of $\omega$? (b) During the sliding, what are the ball’s translational and angular acceleration? (c) How far does the ball slide?

(a) When the ball has slowed down enough to roll smoothly, the linear and angular velocities must be related by the usual formula for smooth rolling: $v = R \omega$.

(b) The ball’s translational acceleration is given by

$$\sum F = ma$$

$$-\mu_k N = -\mu_k mg = ma$$

$$-\mu_k g = a$$
where $N$ is the normal force and $m$ is the mass of the bowling ball. The ball’s angular acceleration is given by

$$\sum \tau = I \alpha$$

$$\mu_k N R = \mu_k m g R = \left[ \frac{2}{5} m R^2 \right] \alpha$$

$$\frac{5 \mu_k g}{2 R} = \alpha$$

Here, $\tau = RF_k \sin \phi$, where $F_k$ is the force of kinetic friction and $\phi$ is the angle between the radius vector to the point where the force of kinetic friction acts and the direction of $F_k$, which is $90^\circ$.

(c) We can find how far the ball slides by first finding $v$ using $a$ and finding $\omega$ using $\alpha$, then finding the time the ball slides by setting $v = R\omega$:

$$v = v_i + at$$

$$= v_i - \mu_k g t_{slide}$$

$$\omega = \omega_i + \alpha t$$

$$= \frac{5 \mu_k g}{2 R} t_{slide}$$

$$v_i - \mu_k g t_{slide} = R \left( \frac{5 \mu_k g}{2 R} t_{slide} \right)$$

$$= (\mu_k g + \frac{5}{2} \mu_k g) t_{slide} = \frac{7}{2} \mu_k g t_{slide}$$

$$t_{slide} = \frac{2 v_i}{7 \mu_k g} = \frac{2}{7} \frac{8.5 \text{ m/s}}{(0.21)(9.8 \text{ m/s}^2)} = 1.18 \text{ s}$$

The distance the ball slides is then given by

$$d = v_i t + \frac{1}{2} a t^2$$

$$= v_i t - \frac{\mu_k g t_{slide}^2}{2}$$

$$= (8.5 \text{ m/s})(1.18 \text{ s}) - \frac{(0.21)(9.8 \text{ m/s}^2)}{2}(1.18 \text{ s})^2 = 8.60 \text{ m}$$

(2b) Two 2.00 kg balls are attached to the ends of a thin rod of negligible mass that is 50.0 cm long. The rod is free to rotate in a vertical plane without friction about a horizontal axis through its centre. With the rod initially horizontal, a 50.0 g wad of putty drops onto one of the balls, hitting it with a speed of 3.00 m/s and sticking to it. (a) What is the angular speed of the system just after the putty hits? (b) What is the ratio of the kinetic energy of the system after the collision to that of the putty just before the collision? (c) Through what angle will the system rotate until it momentarily stops?
(a) The angular momentum of the rod+balls+putty will be conserved in this inelastic collision. Just before the collision, all the angular momentum belongs to the putty. To find this initial angular momentum, we need to choose a point about which we will calculate the angular momentum – I chose the point about which the rod can rotate, i.e., the centre of the rod. The initial angular momentum of the falling putty is then

\[ L_i = r_{\text{putty}}p_{\text{putty}} \sin \phi = r_{\text{putty}}m_{\text{putty}}v_{\text{putty}} \sin \phi \]

\[ = (0.250 \text{ m})(0.0500 \text{ kg})(3.00 \text{ m/s}) \sin(90) \]

\[ = 0.03750 \text{ kg m}^2/\text{s} \]

Here, \( \phi \) is the angle between \( \vec{r} \) and \( \vec{p} \): 

![Diagram](rotation_axis.png)

The angular momentum after the collision will be \( L_f = I \omega \), where \( I \) is the total moment of inertia for the balls and putty. Treating the two balls like point masses:

\[ I_f = 2[M_{\text{ball}}R^2] + m_{\text{putty}}R^2 = [4M_{\text{ball}} + m_{\text{putty}}]R^2 \]

\[ = [4(2.0 \text{ kg}) + 0.050 \text{ kg}](0.250 \text{ m})^2 = 0.503 \text{ kg m}^2 \]

\[ L_i = L_f \]

\[ r_{\text{putty}}m_{\text{putty}}v_{\text{putty}} = I_f \omega_f \]

\[ 0.0375 \text{ kg m}^2/\text{s} = (0.503 \text{ kg m}^2) \omega_f \]

\[ \omega_f = 0.0746 \text{ rad/s} \]

(b) The kinetic energy before the collision is purely due to the putty:

\[ KE_i = \frac{1}{2}m_{\text{putty}}v_{\text{putty}}^2 = \frac{1}{2}(0.050 \text{ kg})(3.00 \text{ m/s})^2 = 0.225 \text{ J} \]

The kinetic energy just after the collision is

\[ KE_f = \frac{1}{2}I \omega^2 = \frac{1}{2}(0.503 \text{ kg m}^2)(0.0746 \text{ rad/s})^2 = 0.00140 \text{ J} \]

Thus, the ratio of the kinetic energies just after and just before the collision is

\[ KE_f/KE_i = 0.00140 \text{ J}/0.225 \text{ J} = 0.00622 \]

In other words, about 99.4% of the initial kinetic energy is lost.

(c) The rod will momentarily stop when it gets to a point where all its initial kinetic energy has been transformed into gravitational potential energy. We can choose any height for \( h = 0 \), so let’s choose the initial horizontal height of the two balls. When the balls are not horizontal, one will have a gravitational potential energy of \( +mg\), while the other will have a gravitational PE of \( mg(-h) = -mgh \). The sum of these two is obviously zero, just as it was before the putty hit. This will always be true - the gravitational potential energies of the two balls will always add to zero. Therefore, it is the mass and height of the putty that will determine where the rod stops. The system will accelerate as it swings downwards, attaining its maximum speed when the putty is at the bottom. The maximum height attained by the putty will be given by

\[ m_{\text{putty}}gh_{\text{max}} = \frac{1}{2}I \omega^2 \]

\[ h_{\text{max}} = \left( \frac{1}{m_{\text{putty}}g} \right) \left( \frac{1}{2}I \omega^2 \right) \]

\[ = \left( \frac{1}{(0.050 \text{ kg})(9.8 \text{ m/s}^2)} \right)(0.00140 \text{ J}) \]

\[ h_{\text{max}} = 0.00286 \text{ m} = 2.86 \text{ mm} \]
Therefore, the angle through which the system swings before it momentarily stops is $180^\circ + \arcsin(0.00286/0.25) = 180.66^\circ$.

(3b) Four identical bricks of length $L$ are stacked on top of one another as shown, such that part of each extends beyond the one beneath. Find in terms of $L$ the maximum values of the overhangs $a_1$, $a_2$, $a_3$, $a_4$, and $h$ such that the stack is in equilibrium.

It may seem odd, but the best way to go about this is to start at the top and work our way down. Looking at the top brick, in order for it to be in equilibrium, there must be no net force and no net torque acting on it. If we imagine starting with the top brick so that its right side is flush with the right side of the brick below, then slowly moving it over to the right, it can remain in equilibrium as long as its centre of mass is located over the brick below - as soon as its centre of mass moves out beyond the edge of the brick below, the top brick will feel a net torque and fall over. Therefore, the maximum distance for $a_1$ is $a_1 = L/2$.

Now look at the top two bricks. They will fall if their centre of mass lies beyond the edge of the next brick down. So let’s find the centre of mass of the two top bricks. It is convenient to choose the right edge of the second brick down as $x = 0$. In this case, the centre of mass of the two bricks will be

$$x_{2,\text{com}} = \frac{m(0) + m(-L/2)}{2m} = -L/4$$

where $m$ is the mass of each of the bricks. The centre of mass of the top two bricks is a distance $L/4$ from the right edge of the second brick down so, and this is the furthest over the bricks can be and be in equilibrium: $a_2 = L/4$.

Now continue with the top three bricks, using the right edge of the third brick down as $x = 0$. The centre of mass of the third brick is located at $x = -L/2$ relative to its right-hand edge. The second brick’s centre of mass is located a distance $L/4$ to the left of the right-hand edge of the third brick, or at $x = -L/4$. The centre of mass of the top brick is located a distance $a_2 = L/4$ to the right of the right-hand edge of the third brick, or at $x = L/4$. Therefore:

$$x_{3,\text{com}} = \frac{m(-L/4) + m(L/4) + m(-L/2)}{3m} = -L/6$$

The centre of mass of the top three bricks is a distance $L/6$ from the right edge of the third brick down so, and this is the furthest over the bricks can be and still be in equilibrium: $a_3 = L/6$.

Carefully carrying out the same procedure for all four bricks yields $a_4 = 1/8$. Thus, the maximum value for $h$ is $h = a_1 + a_2 + a_3 + a_4 = 25L/24$!

Note that it is not necessary to calculate the COM positions for each brick separately as I did above. Once you have the COM of the top two bricks, you can then subsequently treat them as a single object with mass $2m$ sitting at the location of the COM, and so forth.