# Thermal Physics: Problem Set 1

**Due date: 11am, Thursday 31st January 2008**

**Problem 1:** A cinema has $n$ rows of seats. The $i$th row has $M_i$ seats and a total of $N$ cineasts are sitting in the room. The number $N_i \leq M_i$ of people per row can be chosen freely, subject to the overall constraint that

$$\sum_{i=1}^{n} N_i = N.$$

Show that the maximum number of seating arrangements results when the density of people is the same in all rows (i.e. $N_1/M_1 = N_2/M_2 = \cdots = N_n/M_n$).
(Hint: The number of parking arrangements is the product over all levels of the multiplicity function

$$\frac{M_i!}{N_i!(M_i - N_i)!}$$

for each level. Use Stirling's approximation for the factorials and treat the $N_i$ as continuous variables. Use the method of Lagrange multipliers (see below) to find the maximum of the function subject to the constraint.)

**Problem 2:** In the cinema of Problem 1, suppose that it costs $E_i$ to sit in the $i$th row. In addition to the constraint on the total number of people, suppose that we also constrain the total cost $\sum E_i N_i$ of filling the cinema. Show that the maximum number of arrangements (subject now to both constrains) results when

$$\frac{d_i}{1 - d_i} = \exp\left[\frac{\mu - E_i}{\tau}\right], \tag{1}$$

where $d_i = N_i/M_i$ is the density of people in the $i$th row and both $\mu$ and $\tau$ are constants. Thus observe that at low density ($d_i \ll 1$) of people per row falls off exponentially with

the price of sitting there. Show that in general eq.(1) implies that

$$d_i = \frac{1}{\exp\left[\frac{E_i-\mu}{\tau}\right]+1}. \tag{2}$$

Note: The expression on the right hand side of eq. (2) is called the Fermi–Dirac distribution in the statistical mechanics of microphysical systems. Then $d_i$ is interpreted as the probability of occupying a state of energy $E_i$, $\mu$ is called the chemical potential and $\tau$ is proportional to the absolute temperature. In this physical interpretation, the analogue of the constraint on total 'cost' in the above example is the law of conservation of energy.

**The Method of Lagrange Multipliers:** To find an extremum of a function $f(x_1, x_2, \ldots, x_n)$ when no constraint is applied we simply find the solution of the simultaneous equations

$$\frac{\partial f}{\partial x_1} = 0, \qquad \frac{\partial f}{\partial x_2} = 0, \qquad \frac{\partial f}{\partial x_3} = 0, \qquad \text{etc.}$$

To find the extremum of $f(x_1, x_2, \ldots, x_n)$ subject to a constraint on the variables $x_i$ of the form

$$C(x_1, x_2, \ldots, x_n) = 0, \tag{3}$$

we consider the function

$$G(x_1, x_2, \ldots, x_n; \lambda) = f(x_1, x_2, \ldots, x_n) + \lambda C(x_1, x_2, \ldots, x_n). \tag{4}$$

Notice that $G = f$ when the constraint eq. (3) is satisfied. To find the maximum of $f$ subject to the constraint we then find the solution of the equations

$$\frac{\partial G}{\partial x_1} = 0, \qquad \frac{\partial G}{\partial x_2} = 0, \qquad \frac{\partial G}{\partial x_3} = 0, \qquad \text{etc.}$$

along with the additional equation

$$\frac{\partial G}{\partial \lambda} = 0 \tag{5}$$

which ensures that the constraint condition eq. (3) is satisfied.

The auxiliary variable in eq. (4) is called a Lagrange multiplier. Usually one does not bother actually solving eq. (5) directly. One just solves the other equations $\frac{\partial G}{\partial x_i} = 0$, for $i = \{1, n\}$, treating $\lambda$ as an unknown constant. Then, if necessary, one finds a value of $\lambda$ which will make the solution found for the $x_1$ satisfy the constraint in eq. (3).
If there is more than one constraint (as in Problem 2 above), one introduces a separate Lagrange multiplier for each constraint – hence the presence of two constants $\mu$ and $\tau$ in eq. (1)