Hot Electron Transport

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Overview

• Introduction
• Basic transport concepts
• High field phenomena
• Scattering mechanisms
• The Boltzmann transport equation
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Introduction

- The lattice temperature $T_0$
- Electrons in thermal equilibrium
- Hot electrons
- Scope of this talk
Temperature is a measure of the average energy of a quantum of the system.

- Lattice vibrations are quantised as phonons.
- The average energy of a phonon determines the lattice temperature $T_0$.

Schematic of a crystal lattice in its equilibrium configuration... and with a (longitudinal) phonon mode...
Electrons in thermal equilibrium

The Fermi-Dirac distribution

\[ f_0(\epsilon) = \frac{1}{1 + \exp \left( \frac{\epsilon - \epsilon_F}{k_B T_0} \right)} \]

Energetic distribution of electrons in thermal equilibrium with the crystal lattice

\[ k_B T_0 / \epsilon_F = 0 \]

\[ k_B T_0 / \epsilon_F = 0.2 \]

\[ k_B T_0 / \epsilon_F = 0.1 \]
Hot electrons

- The term **hot electrons** refers to a population of high-energy electrons out of thermal equilibrium with the crystal lattice.
- Hot electrons may be sometimes be characterised by an electron temperature $T_e$, such that $T_e > T_0$.
- Hot electrons are typically produced by high electric fields, driving the electrons to higher energies.
- Materials with a population of hot electrons may exhibit non-linear effects, such as
  - Electrical breakdown
  - Negative differential resistance
The purpose of this talk is:

- To introduce the audience to the concept of hot electrons and some high-field phenomena
- To give the audience an introduction into the theoretical approaches used to understand the details of hot electron transport

Limitations of the talk:

- We shall only be looking at the semi-classical approach, so there will be no discussion of quantum transport or non-equilibrium Green’s functions
- Due to time constraints, we shall only consider the solution of Boltzmann’s transport equation
- Although hot electrons are particularly important in nano-devices, there will be no specialisation to low dimensional structures
• Introduction
• Basic transport concepts
• High field phenomena
• Scattering mechanisms
• The Boltzmann transport equation
Basic transport concepts

- Ballistic transport
- Energy and momentum relaxation
- Describing energy bands
- Group velocity and density of states
- The non-equilibrium distribution function
- Transport properties
- The conservation equations
Ballistic transport describes a regime of transport in which no scattering (and hence no energy or momentum relaxation) takes place.

- The application of an electric field $\mathbf{E}$ gives rise to a spatially varying potential $V(r)$.
- If the electron travels ballistically for a distance $\Delta x$, it gains a kinetic energy $eE\Delta x$ above the conduction band edge.
- If $eE\Delta x$ is significantly greater than the thermal energy $k_B T_0$, then we may describe the electron as being hot.
An electron will not continue to travel ballistically indefinitely. At some point it is likely to scatter, which will change its momentum and, sometimes, its energy.

- The electron undergoes an energy relaxing (inelastic) scattering event, changing to a state with energy $\epsilon'$.
- We may define an energy relaxation time $\tau_\epsilon(\epsilon)$, which is the average time an electron of energy $\epsilon$ will travel before undergoing an inelastic scattering event.
- Note that since $e = \hbar \omega$, $\tau_\epsilon(\epsilon)$ may also be thought of as a coherence time.
Momentum relaxation

- The electron undergoes a momentum scattering event, changing its momentum from $\hbar k$ to $\hbar k'$
- If $|k| = |k'|$, then energy is conserved and the interaction is elastic
- We may define an energy relaxation time $\tau_m(\epsilon_k)$, which is the average time an electron of energy $\epsilon_k$ and momentum $\hbar k$ will travel before undergoing a momentum changing scattering event
- Note that, particularly at low $T_0$, we usually have $\tau_\epsilon(\epsilon) >> \tau_m(\epsilon_k)$, meaning an electron may change momentum many times before losing coherence
Fig. 1 (a) The Brillouin zone of a face-centred cubic crystal, showing the \( \Gamma \), \( X \) and \( L \) symmetry points. (b) Schematic band-structure of a direct band-gap semiconductor showing the \( \Gamma \), \( X \) and \( L \) valleys.
Surfaces of constant energy in the Brillouin zone

Spherical valley
(Here centred on \( \Gamma \) point)

Spheroidal valley
(Here lying along the \( \Delta \) line)
The effective mass $m^*$ is inversely proportional to the curvature of the band ($d^2\varepsilon/dk^2$) at the band edge.

\[ \varepsilon(k) = \frac{\hbar^2 k^2}{2m^*} \]

\[ \gamma(\varepsilon) = \frac{\hbar^2 k^2}{2m^*} \]
The $\gamma$ function describing the dispersion relations modifies:

**The electron group velocity**
(here just the energy dependent magnitude is given)

$$v(\epsilon) = \left( \frac{2\gamma(\epsilon)}{m^*} \right)^{1/2} \left( \frac{d\gamma}{d\epsilon} \right)^{-1}$$

**The density of states (DOS)**

$$D(\epsilon) = V_C \frac{(2m^*)^{3/2}}{4\pi^2\hbar^3} \gamma(\epsilon)^{1/2} \frac{d\gamma}{d\epsilon}$$
Types of scattering

inter-valley

intra-valley

X

Γ

L
• The Fermi-Dirac distribution $f_0(\epsilon)$ gives the energetic electron distribution function in equilibrium.

• We assume that under non-equilibrium conditions, there exists a distribution function $f(k)$, giving the probability a state in $k$-space near to $k$ is occupied.

• This is the basis of the semi-classical approach.

• The principal task of semi-classical transport calculations is to find $f(k)$.
The **current density** is the integral of the group velocity times the probability of occupation, integrated over all states.

\[ j = -\frac{2e}{(2\pi)^3} \int v(k) f(k) \, d^3k \]

Note that phenomenologically:

\[ j = \sigma(E)E \]

where \( \sigma \) is the conductivity tensor and \( E \) must now be embedded in \( f(k) \) somehow.

Note also that when the integral above is turned into an integral over energy, because \( v(\epsilon)D(\epsilon) \sim m^* \), the conductivity must be (roughly) proportional to the effective mass.
The mobility is defined as

\[
\mu(E) = \frac{\sigma(E)}{en}
\]

where the **free carrier density** is given by

\[
n = \frac{2}{(2\pi)^3} \int_{CB} f(k) \, d^3k
\]

The current density can then be written

\[
j = ne\mu(E)E = nev_D(E)
\]

where \(v_D\) is the drift velocity (an average quantity)

Note that the mobility will also be proportional to the effective mass
The conservation equations

Balance of energy:

\[ \frac{d\epsilon}{dt} = -eE \cdot v_D - \left\langle \frac{d\epsilon}{dt} \right\rangle_s \]

Balance of momentum:

\[ \frac{d\hbar k}{dt} = -eE - \left\langle \frac{d\hbar k}{dt} \right\rangle_s \]

Rates of change due to scattering, averaged over \( f(k) \)
• Introduction
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• **High field phenomena**
• Scattering mechanisms
• The Boltzmann transport equation
High-field phenomena

- Avalanche multiplication
- Negative differential resistance
- Microwave oscillations
Impact ionization

Schematic of impact ionization for electrons

(a) An electron gains an energy $eF\Delta x$ ($> E_g$) from the field.

(b) After colliding with an ion, the electron gives up its energy to form an electron-hole pair.

N.B. This is the inverse process to Auger recombination
We can define the probability that a given carrier will ionize an electron-hole pair in distance $dx$ by $\alpha(E)dx$ (for electrons) or $\beta(E)dx$ (for holes), where

- $\alpha(E)$ is the **electron ionization coefficient** and
- $\beta(E)$ is the **hole ionization coefficient**

Here, $\alpha(E)$ and $\beta(E)$ are interpreted as *rates* (per unit distance). Alternatively, we can think in terms of the quantities

- $1/\alpha(E)$ the average distance an electron travels between ionizing collisions and
- $1/\beta(E)$ the average distance a hole travels between ionizing collisions.
Simple model of avalanche breakdown

\[ n_e \hat{v}_e + \delta x \hat{v}_e = n_e \hat{v}_e \]
\[ + \alpha n_e \hat{v}_e + \beta n_h \hat{v}_h + \delta x \hat{v}_h \delta x \]
\[ n_h \hat{v}_h = n_h \hat{v}_h \]
\[ + \alpha n_e \hat{v}_e + \beta n_h \hat{v}_h + \delta x \hat{v}_h \delta x \]

The \( v_e, v_h \) are the drift velocities

\[ \frac{dj_e}{dx} = \alpha j_e \hat{v}_e + \beta j_h \]
\[ \frac{dj_h}{dx} = -\alpha j_e \hat{v}_e - \beta j_h \]
Example solution of avalanche breakdown

Solving the differential equations

\[
\frac{dj_e}{dx} = \alpha j_e - \beta j_h \\
\frac{dj_h}{dx} = -\alpha j_e - \beta j_h
\]

leads to

\[
\frac{J}{J_0} = \frac{e^{-\beta x} \alpha - \beta e^{-\beta x}}{\alpha - \beta e^{-\beta x}}
\]

with boundary conditions

\[
\begin{align*}
j_e(x=0) &= J_0 \\
j_e(x) &= J \\
j_h(x) &= 0
\end{align*}
\]

This solution becomes infinite when

\[
\alpha = \beta e^{-\beta A x}
\]

implying avalanche breakdown

Avalanche multiplication is exploited as a gain mechanism in avalanche photodiodes (APDs)
Ionization coefficients for different materials

![Graph showing ionization coefficients for different materials.](image)

Negative differential resistance

Current ($\sim V_D$) decreases with increasing voltage.

Ohmic region (current proportional to voltage)

Voltage ($\sim E$)
Negative differential resistance

Differential resistance: $R = \frac{dV}{dl}$

Such an NDR and the consequent microwave oscillations observed can be explained by the Ridley-Watkins-Hilsum mechanism.
Transferred electrons

$m_1^*$ small, so $\mu_1$ high

$m_2^*$ large, so $\mu_2$ low

Inter-valley scattering

$\Delta \epsilon$

 Carrier fluctuations

potential due to uniform field

net potential

space-charge potential due to carrier density fluctuation

Carrier fluctuations

Direction of travel for electrons

Field reduced in trailing edge

Field increased in leading edge
Positive $dl/dV$

Under normal conditions of positive differential resistance, there is a net current out of the space-charge fluctuation.
Positive $dl/dV$

Direction of travel for electrons

Current into fluctuation reduced

Current out of fluctuation increased

And the fluctuation is therefore smoothed away
In the case of negative differential resistance, the reverse situation pertains and we have a net flow of charge into the fluctuation.
Hence the space charge increases until the currents in and out of the fluctuation equalise.
The space-charge domain will reach a steady configuration when the currents at the leading and trailing edges are equal.

By thermodynamic considerations, the configuration that minimises the current is preferred.

The high-field domain travels along the field with the drift velocity $V_D$ associated with $I_1$. For short devices, there is likely to be only one domain in the device at a time.
As the domain leaves the device at the positive terminal,
another is nucleated at the negative terminal.

Hence we obtain current oscillations with a frequency of approximately $V_D/L$. This is the mechanism of the **Gunn diode**.

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• **Scattering mechanisms**
• The Boltzmann transport equation
Scattering mechanisms

- The intrinsic scattering rate
- Phonons
- The Coulomb interaction
- Other scattering mechanisms
The intrinsic scattering rate

\[
s(k', k) = \frac{2\pi}{\hbar} |\langle k | V | k' \rangle|^2 \frac{V_C}{(2\pi)^3} \delta (\epsilon' - \epsilon)
\]

Scattering rate per \( k \)-state from \( k' \) to \( k \)

Matrix element for the transition probability per unit time from state \( k' \) to \( k \)

\( V_C \) is the volume of the crystal; \( 2\pi^3 / V_C \) is the volume in reciprocal space occupied by a \( k \)-state

Here, the Dirac delta gives the condition for conservation of energy (in this case, for elastic scattering)
Acoustic phonons are vibrational displacements of the unit cells (with frequencies in the acoustic range).
Optical phonons are vibrational displacements basis atoms (with frequencies in the optical range).

- Transverse (TO) modes
- Longitudinal (LO) modes
If the basis atoms are of a different type, there may be a redistribution of charge between them…

… giving rise to an electric dipole $\mathbf{p}$ between them

Longitudinal polar optical phonons couple very strongly with electrons
Note that the frequency of the optical phonons is roughly constant, whilst, near the \( \Gamma \)-point, the frequency of acoustic phonons is proportional to wavevector.
For phonons, the scattering is inelastic. The rate must also be decomposed into terms for emission and absorption of a phonon with energy $\hbar \omega$

$$s(k', k) = s_A(k', k) \delta (\epsilon' - \epsilon + \hbar \omega) + s_E(k', k) \delta (\epsilon' - \epsilon - \hbar \omega)$$

where

$$s_A(k', k) = \frac{2\pi}{\hbar} \left| M^{n'n'}_{kk'} - 1 \right|^2 \frac{V_C}{(2\pi)^3}$$

is for absorption of a phonon and

$$s_E(k', k) = \frac{2\pi}{\hbar} \left| M^{n'n'+1}_{kk'} \right|^2 \frac{V_C}{(2\pi)^3}$$

is for emission.

$n$ is the phonon occupation number (given by the Bose-Einstein factor).
It is known that a bare Coulomb potential leads to electron scattering at all distances.

\[ V(r) = \frac{Ze}{4\pi \varepsilon |r - R|} \]

However, the redistribution of free charge around a charged particle usually gives rise to some kind of screening (here \(q_0\) is the reciprocal screening length).
Scattering processes involving the Coulomb interaction

- Charged impurity scattering
  - Ionized donors and acceptors
  - Significant at low $T_0$

- Electron-hole scattering
  - Significant for narrow gap semiconductors at room temperature

- Electron-electron scattering
  - Does not relax net momentum but may randomize momentum, with consequences for the high-field distribution function

- Impact-ionization
  - Significant at high fields once free electrons attain sufficient energy to ionize a valence-band electron
Other scattering mechanisms

• Neutral impurity scattering
  – For non-ionized donors and acceptors, only significant at very low $T_0$
  – More generally, may be contribute to resistivity over all temperature ranges
  – Resonant states (such as N in GaAs) may give very strong scattering

• Alloy scattering
  – Arises from random deviations from the averaged potential due to substitutional atomic species in alloys
  – Can provide a mechanism of inter-valley scattering
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• High field devices
The Boltzmann transport equation

- General form of the BTE
- The linearized distribution function
- Low-field solution and the ladder method
- High-field solution for a single valley
- Inter-valley transfer
General form of the BTE

\[
\frac{df(k)}{dt} = \left( \frac{\partial f(k)}{\partial t} \right)_s - \frac{dk}{dt} \cdot \nabla_k f(k) - \mathbf{v}(k) \cdot \nabla f(k)
\]

Temporal rate of change

Force term

Scattering term

Density / temperature gradient term

Ludwig Boltzmann (1844 – 1906)
The scattering integral

\[
\left( \frac{\partial f(k)}{\partial t} \right)_s = \int s(k', k)f(k') \left[ 1 - f(k) \right] - s(k, k')f(k) \left[ 1 - f(k') \right] \, d^3k'
\]

- **probability** \( k' \) **not occupied**
- **probability** \( k \) **occupied**
- **probability** \( k' \) **occupied**
- **probability** \( k' \) **not occupied**
- **intrinsic scattering** from \( k \) to \( k' \)
- **probability** \( k' \) **occupied**
- **probability** \( k \) **not occupied**

**COST ACTION**
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Hot Electron Transport
M.P. Vaughan
• Put $d\mathbf{k}/dt = \mathbf{F}/\hbar$, where $\mathbf{F}$ is the applied force

• Assume no concentration / temperature gradient

• Assume steady state, so $df(k)/dt = 0$

Then we are left with

$$\frac{\mathbf{F}}{\hbar} \cdot \nabla_{\mathbf{k}} f(k) = \left( \frac{\partial f(k)}{\partial t} \right)_s$$

We shall take $\mathbf{F} = -e\mathbf{E}$, where $\mathbf{E}$ is the electric field
The linearized distribution function

(1) Expand \( f(k) \) to first order
\[
f(k) = f_0(k - \delta k) = f_0(\epsilon) - \nabla_k f_0(\epsilon) \cdot \delta k
\]

(2) Use change rule and \( \mathbf{v} = \nabla_k \omega = \nabla_k \epsilon / \hbar \) for group velocity
\[
f(k) = f_0(\epsilon) - \hbar \frac{df_0(\epsilon)}{d\epsilon} \mathbf{v}(k) \cdot \delta k
\]

(3) Put \( \hbar \delta k \) as momentum gained from \( \mathbf{E} \) before scattering
\[
\delta k = -\frac{e \mathbf{E} \tau(\epsilon)}{\hbar}
\]

(4) Substitute in \( \delta k \)
\[
f(k) = f_0(\epsilon) + e \tau(\epsilon) \frac{df_0(\epsilon)}{d\epsilon} \mathbf{v}(k) \cdot \mathbf{E}
\]

(5) Define \( f_1 \) as asymmetric part of \( f(k) \)
\[
f_1(\epsilon) = e \tau(\epsilon) \frac{df_0(\epsilon)}{d\epsilon} E \nu(\epsilon) \chi
\]

Our task now is to determine \( \tau(\epsilon) \)
Inserting

\[ f(\mathbf{k}) = f_0(\epsilon) + e\tau(\epsilon)\frac{df_0(\epsilon)}{d\epsilon} \mathbf{v}(\mathbf{k}) \cdot \mathbf{E} \]

into

\[ j = -\frac{2e}{(2\pi)^3} \int \mathbf{v}(\mathbf{k}) f(\mathbf{k}) \, d^3\mathbf{k} \]

Recalling that the symmetric part of \( f(\mathbf{k}) \) disappears due to odd parity of \( \mathbf{v}(\mathbf{k}) \)

leads to

\[ j = \sigma(\mathbf{E}) \mathbf{E} \]

\[ \sigma = -\frac{2e^2}{3(2\pi)^3} \int v^2(\epsilon_k) \tau(\epsilon_k) \frac{df_0(\epsilon_k)}{d\epsilon} \, d^3\mathbf{k} \]

for the low-field conductivity.
Using Boltzmann’s equation in the steady state

\[ \frac{F}{\hbar} \cdot \nabla_k f(k) = \left( \frac{\partial f(k)}{\partial t} \right)_s \]

neglecting terms in \( E^2 \) the left-hand-side becomes

\[ -\frac{e}{\hbar} \mathbf{E} \cdot \nabla_k f(k) = -e \mathbf{E} \cdot \mathbf{v}(k) \frac{df_0(\epsilon)}{d\epsilon} = -eE v(\epsilon) x \frac{df_0(\epsilon)}{d\epsilon} \]

and the right-hand-side is decomposed into zero and first-order scattering terms

\[ \left( \frac{\partial f(k)}{\partial t} \right)_s = \left( \frac{\partial f_0(\epsilon)}{\partial t} \right)_s + \left( \frac{\partial f_1(\epsilon)}{\partial t} \right)_s \]
Low-field scattering integral (1)

For low-field $f_0(\epsilon)$ is the Fermi-Dirac factor (i.e. equilibrium distribution), hence

$$\left( \frac{\partial f_0(\epsilon)}{\partial t} \right)_s = \int s(k', k) f_0(\epsilon') [1 - f_0(\epsilon)] - s(k, k') f_0(\epsilon) [1 - f_0(\epsilon')] \, d^3k'$$

disappears, being an expression of the principle of detailed balance which just leaves

$$\left( \frac{\partial f_1(\epsilon)}{\partial t} \right)_s = \int s(k', k) \{ f_1(\epsilon') [1 - f_0(\epsilon)] - f_1(\epsilon) f_0(\epsilon') \} - s(k, k') \{ f_1(\epsilon) [1 - f_0(\epsilon')] - f_1(\epsilon') f_0(\epsilon) \} \, d^3k'$$

on the right-hand-side.
Substituting for \( f_1(\epsilon) \), the scattering integral becomes

\[
\left( \frac{\partial f(k)}{\partial t} \right)_s = -e \frac{df_0(\epsilon)}{d\epsilon} E v(\epsilon) x
\]

\[
\times \int s(k', k) \frac{f_0(\epsilon')}{f_0(\epsilon)} \left\{ \tau(\epsilon) - \tau(\epsilon') \frac{v(\epsilon') x'}{v(\epsilon) x} \right\} d^3 k'
\]

The factor of \( x'/x \) can be dealt with by noting that

\[
\hat{E} \cdot \hat{k}' = \cos \theta' = \cos \theta \cos \alpha' + \sin \theta \sin \alpha' \cos \phi
\]

where \( \alpha' \) is the angle between \( k \) and \( k' \). Since the second term disappears on integration, \( x'/x \) is effectively equal to \( \cos \alpha' \).
On comparison of the left and right-hand-sides of the Boltzmann equation, we find

\[ \int s(k', k) \frac{f_0(\epsilon')}{f_0(\epsilon)} \left\{ \tau(\epsilon) - \tau(\epsilon') \frac{v(\epsilon')}{v(\epsilon)} \cos \alpha' \right\} d^3k' = 1 \]

Note that we also have

\[ \left( \frac{\partial f_1(\epsilon)}{\partial t} \right)_s = -e \frac{df_0(\epsilon)}{d\epsilon} Ev(\epsilon)x = -\frac{f_1(\epsilon)}{\tau(\epsilon)} \]

where the \( \tau(\epsilon) \) are to be calculated using the top expression.
Putting

\[ s(k', k) = s_e(k', k) \delta(\epsilon' - \epsilon) \]

for a general elastic process, the \( \tau(\epsilon) \) become equal and may be taken outside the integral, whilst the \( f_0(\epsilon) \) and \( v(\epsilon) \) cancel out, leaving

\[ \int s_e(k', k) \{1 - \cos \alpha\} \delta(\epsilon' - \epsilon) \, d^3k' = w_e(\epsilon) = \frac{1}{\tau_m(\epsilon)} \]

which is the definition of a momentum relaxation time \( \tau_m(\epsilon) \).

Using the general expression for the intrinsic scattering rate, we have

\[ w_e(\epsilon) = \frac{2\pi}{\hbar} |\langle k|V|k'\rangle|^2 V_C \int \delta(\epsilon' - \epsilon) \frac{d^3k'}{(2\pi)^3} \]

\[ = \frac{2\pi}{\hbar} |\langle k|V|k'\rangle|^2 D(\epsilon) \]

i.e. Fermi’s Golden Rule.
In the case of phonon scattering we can put

\[ s(k', k) = s_A(k', k)\delta (\epsilon' - \epsilon + \hbar\omega) + s_E(k', k)\delta (\epsilon' - \epsilon - \hbar\omega) \]

When this is inserted into the scattering integral, the effect of the Dirac delta is to bring out the \( \tau(\epsilon) \) at different energies:

\[ A(\epsilon)\tau(\epsilon - \hbar\omega) + B(\epsilon)\tau(\epsilon) + C(\epsilon)\tau(\epsilon + \hbar\omega) = 1 \]

where the \( A(\epsilon), B(\epsilon) \) and \( C(\epsilon) \) are the new scattering integrals.

For acoustic (deformation potential) and non-polar optical phonon scattering, the \( A(\epsilon) \) and \( C(\epsilon) \) disappear, giving the normal relaxation time.

*However*, this is **not** the case for polar optical phonon scattering!
For polar optical phonon scattering we have

\[ A(\epsilon)\tau(\epsilon - \hbar \omega) + B(\epsilon)\tau(\epsilon) + C(\epsilon)\tau(\epsilon + \hbar \omega) = 1 \]

where the \( A(\epsilon), B(\epsilon) \) and \( C(\epsilon) \) do not disappear.

Hence, \( \tau(\epsilon) \) at any particular energy becomes coupled to \( \tau(\epsilon + \hbar \omega) \) and \( \tau(\epsilon - \hbar \omega) \).

This suggests the picture of a phonon ladder, with rungs \( \hbar \omega \) apart, where the scattering at any rung is coupled to the rungs above and below it.
Example of the polar optical phonon scattering relaxation time (for GaAs). The black line is the calculation with the number of rungs truncated to 6.
The distribution function may be expanded as a series in the Legendre polynomials $P_j(x)$

$$f(k) = \sum_j h_j(\epsilon) P_j(x)$$

Plot of the first six Legendre polynomials

Note that the even orders are symmetric in $x \cos(\theta)$ whilst the odd orders are antisymmetric.
Inserting this into the left-hand-side of Boltzmann’s equation gives for the force term

\[-\frac{e}{\hbar} \mathbf{E} \cdot \nabla_k f(k) = -eE \sum_j \left\{ \frac{j + 1}{2j + 1} \left[ v(\epsilon) \frac{d h_j}{d\epsilon} - j \frac{h_j(\epsilon)}{\hbar k_c} \right] P_{j+1}(x) \right\}

\[+ \frac{j}{2j + 1} \left[ v(\epsilon) \frac{d h_j}{d\epsilon} + (j + 1) \frac{h_j(\epsilon)}{\hbar k_c} \right] P_{j-1}(x) \]
Limiting this to terms up to $j=2$, this is

$$\frac{e}{\hbar} \mathbf{E} \cdot \nabla_k f(k) = -eE \left\{ \frac{1}{3} \left[ v(\epsilon) \frac{dh_1}{d\epsilon} + 2 \frac{h_1(\epsilon)}{\hbar k} \right] P_0(x) + v(\epsilon) \frac{dh_0}{d\epsilon} P_1(x) \right. \\
+ \left. \frac{2}{3} \left[ v(\epsilon) \frac{dh_1}{d\epsilon} - \frac{h_1(\epsilon)}{\hbar k} \right] P_2(x) \right\}$$
Decomposing the scattering integral into zero and first order integrals, we have

\[
\left( \frac{\partial f(k)}{\partial t} \right)_s = \left( \frac{\partial h_0(\epsilon)}{\partial t} \right)_s + \left( \frac{\partial h_1(\epsilon) x}{\partial t} \right)_s
\]

where the zero-order integral is

\[
P_0(x)I_0 = \int s(k', k) h_0(\epsilon') [1 - h_0(\epsilon)] - s(k, k') h_0(\epsilon) [1 - h_0(\epsilon')] \ d^3k'
\]

and the first-order integral is

\[
P_1(x)I_1 = x \int s(k', k) \left\{ h_1(\epsilon') \cos \alpha [1 - h_0(\epsilon)] - h_1(\epsilon) h_0(\epsilon') \right\}
\]

\[ - s(k, k') \left\{ h_1(\epsilon) [1 - h_0(\epsilon')] - h_1(\epsilon') \cos \alpha' h_0(\epsilon) \right\} \ d^3k'
\]
High-field solution

Equating the coefficients of the Legendre polynomials

\[-\frac{eE}{3} \left[ v(\epsilon) \frac{dh_1}{d\epsilon} + 2 \frac{h_1(\epsilon)}{\hbar k} \right] = I_0\]

\[-eE v(\epsilon) \frac{dh_0}{d\epsilon} = I_1\]

or, in terms of the $\gamma(\epsilon)$ function and its derivative

\[-\frac{eE}{3} \left( \frac{2}{m^* \gamma(\epsilon)} \right)^{1/2} \left( \frac{d\gamma}{d\epsilon} \right)^{-1} \frac{d}{d\epsilon} [\gamma(\epsilon) h_1(\epsilon)] = I_0\]

\[-eE \left( \frac{2\gamma(\epsilon)}{m^*} \right)^{1/2} \left( \frac{d\gamma}{d\epsilon} \right)^{-1} \frac{dh_0}{d\epsilon} = I_1\]
Using the low-field result for the derivative of the first-order term in $f(k)$ due to scattering

$$
\frac{1}{x} \left( \frac{dh_1(\epsilon)x}{dt} \right)_s = -\frac{h_1(\epsilon)}{\tau(\epsilon)}
$$

we obtain the differential equation

$$
-\frac{2e^2E^2}{3m^*} \left( \frac{1}{\gamma(\epsilon)} \right)^{1/2} \left( \frac{d\gamma(\epsilon)}{d\epsilon} \right)^{-1} \frac{d}{d\epsilon} \left[ \gamma^{3/2}(\epsilon)\tau(\epsilon) \frac{dh_0}{d\epsilon} \right] = I_0
$$

Note that the $\pi(\epsilon)$ are obtained from $I_1$. The solution of the above equation is the electric field dependent part of $f(k)$, $h_0(\epsilon)$. 
General overview:

To deal with inter-valley transfer, it is useful to decompose the distribution function into the distributions in each valley.

Each valley will have associated with it a differential equation like that just derived for the distribution function.

However, we must also bring in a scattering integral due to inter-valley transfer.

These additional terms will couple the differential equations together, which must then be solved simultaneously.
Other semi-classical approaches

• The drifted Maxwellian approach
  – The distribution function is assumed to be a Maxwellian shifted by an average drift k-vector
  – Validity based on assumption that momentum has been randomised by electron-electron scattering
  – Introduces the electron temperature $T_e$ into the formulation explicitly

• Monte-Carlo simulation
  – Monte Carlo techniques used to find the distribution function directly
  – Employs a method of mapping a uniform random distribution into scattering probabilities
  – If implemented correctly, can give very realistic results by reducing modelling assumptions
- Introduction
- Basic transport concepts
- High field phenomena
- Scattering mechanisms
- The Boltzmann transport equation
Further reading

The Ridley-Watkins-Hilsum mechanism


The ladder method for polar-optical phonon scattering


High-field solution of the Boltzmann equation


Recommended text book