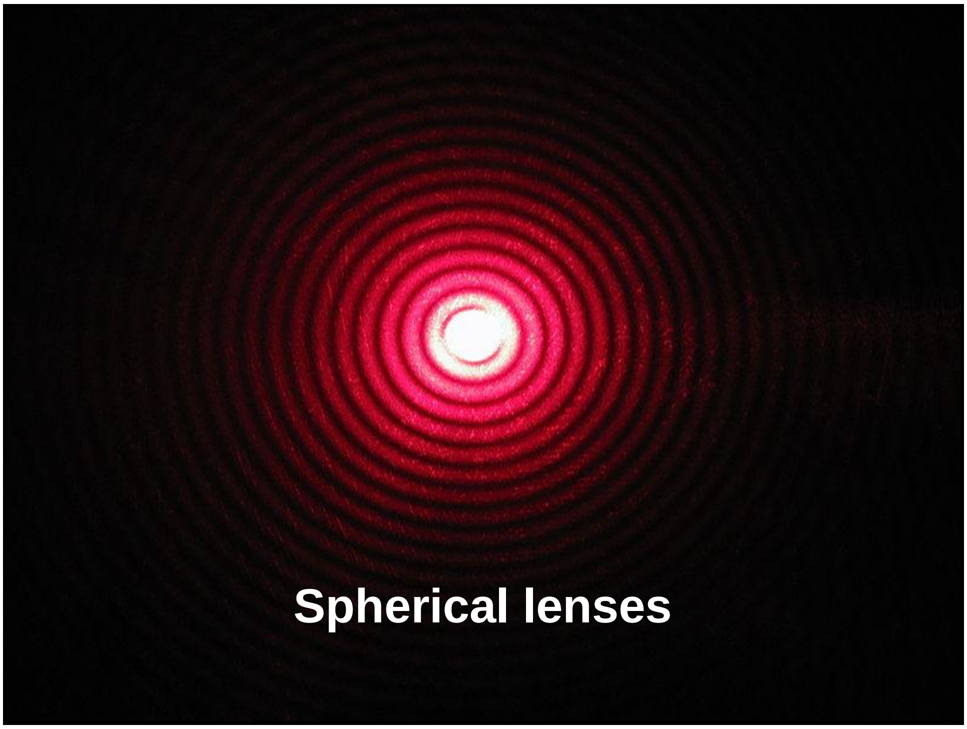
PY3101 Optics

Spherical Lenses and Mirrors

M.P. Vaughan

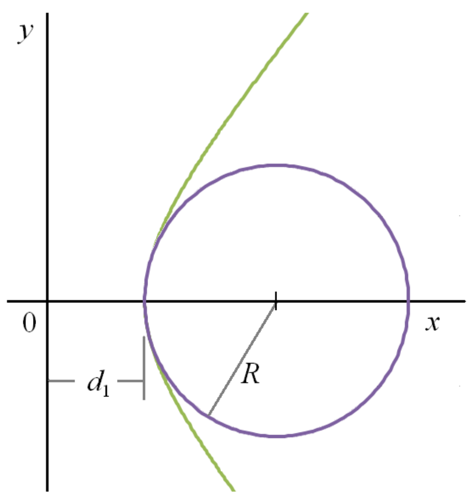
Learning objectives

- Analysis of spherical lenses
- Analysis of ‘thin’ lenses
  - Thin lens equation
  - Lens maker’s formula
  - Gaussian lens formula
- Lens and mirror sign conventions
- Thin lenses in combination.
- Analysis of spherical mirrors
- Image construction
  - Magnification

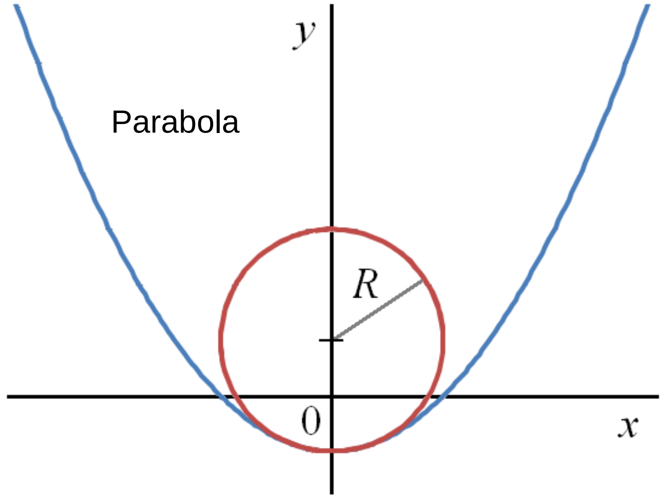


Spherical approximation of perfect imaging

Hyperbola

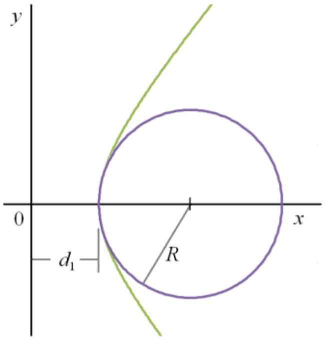


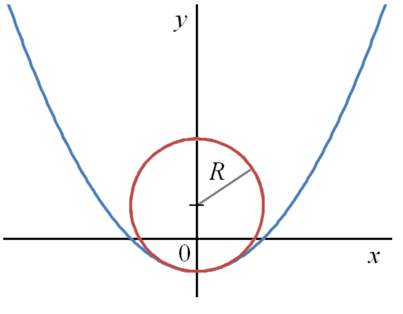
Spherical approximation of perfect imaging



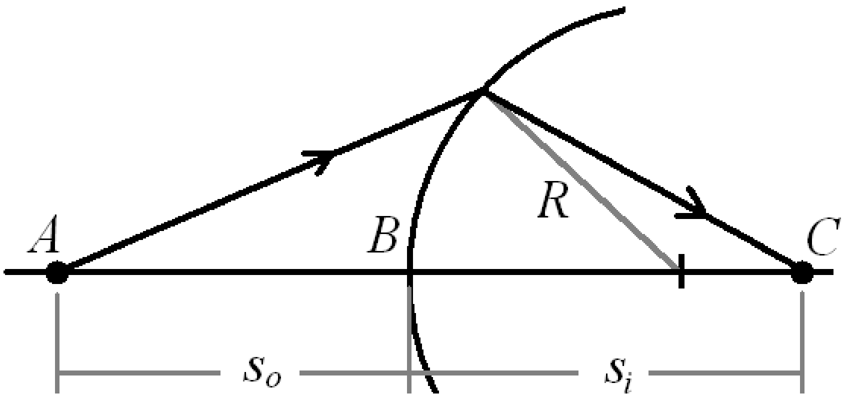
Imaging by a spherical lens

- Cannot obtain perfect imaging
- Reasonable approximation possible



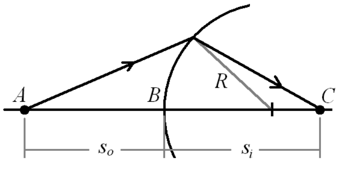


Lens sign conventions



Lens sign conventions

- Light is always taken to propagate from left to right.
- If A is to the left of B, then s_o is taken to be positive (and vice versa).
- If C is to the right of B, then s_i is taken to be positive (and vice versa).

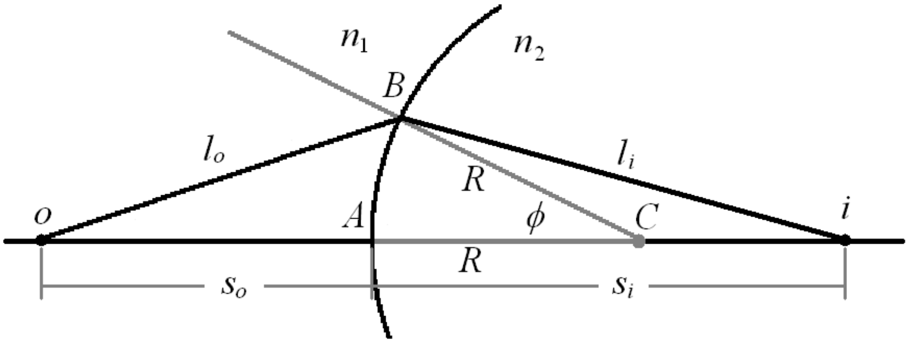


- If the centre of the sphere is to the right of B, R is taken to be positive. This is a convex lens.
- If the centre of the sphere is to the left of B, R is taken to be negative. This is a concave lens.

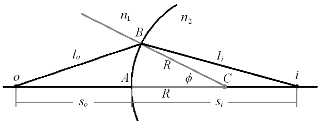
Imaging by a spherical lens

- **Requirement for perfect imaging is that all the rays have equal optical path-length.**
- **That is, we require Λ to be a constant**

Imaging by a spherical lens



The optical path length



Using the cosine rule

$$l_o^2 = (s_o + R)^2 + R^2 - 2(s_o + R)R\cos\phi$$

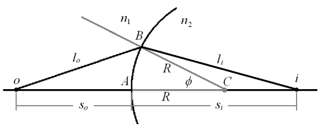
and

$$l_i^2 = (s_i - R)^2 + R^2 + 2(s_i - R)R\cos\phi,$$

whilst

$$\Lambda = n_1 l_o + n_2 l_i.$$

The optical path length



We wish to have

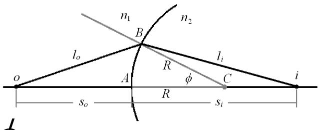
$$\frac{d\Lambda}{d\phi} = 0.$$

So

$$\frac{d\Lambda}{d\phi} = n_1 \frac{dl_o}{d\phi} + n_2 \frac{dl_i}{d\phi} = 0.$$

The optical path length

Now



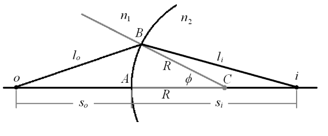
$$\frac{dl_o}{d\phi} = \frac{(s_o + R)R\sin\phi}{\left[(s_o + R)^2 + R^2 - 2(s_o + R)R\cos\phi\right]^{1/2}},$$

$$= \frac{(s_o + R)R\sin\phi}{l_o}.$$

Similarly

$$\frac{dl_i}{d\phi} = -\frac{(s_i - R)R\sin\phi}{l_i}.$$

The optical path length



Thus, inserting into

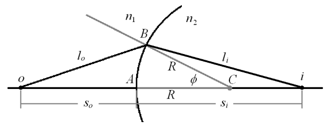
$$\frac{d\Lambda}{d\phi} = n_1\frac{dl_o}{d\phi} + n_2\frac{dl_i}{d\phi} = 0,$$

we have

$$n_1\frac{(s_o + R)}{l_o} = n_2\frac{(s_i - R)}{l_i}.$$

The optical path length

This may be rearranged to give



$$\boxed{\left(\frac{n_1}{l_o}+\frac{n_2}{l_i}\right)R=\frac{n_2 s_i}{l_i}-\frac{n_1 s_o}{l_o}.}$$

However

$$l_o = l_o(\phi, s_o),$$
$$l_i = l_i(\phi, s_i),$$

so no closed form solutions.

The paraxial approximation

Small angle approximation

$$\sin\phi \approx \phi, \quad \cos\phi \approx 1.$$

So

$$l_o^2 \to (s_o + R)^2 + R^2 - 2(s_o + R)R = s_o^2$$

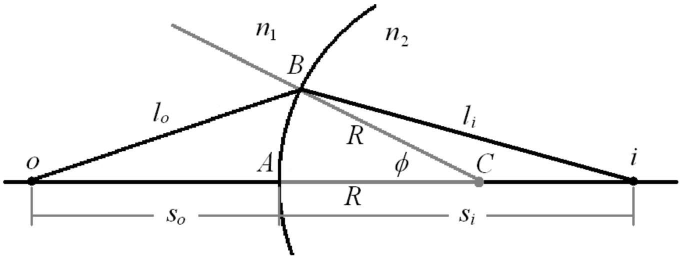
and

$$l_i^2 \to (s_i - R)^2 + R^2 + 2(s_i - R)R = s_i^2.$$

The paraxial approximation

With these approximations, we obtain

$$\boxed{\frac{n_1}{s_o}+\frac{n_2}{s_i}=\frac{1}{R}(n_2-n_1).}$$



The focal length

In the case

$$s_0 \to \infty,$$

$$\frac{n_2}{s_i}=\frac{1}{R}(n_2-n_1).$$

From this, we may define the *focal length within* the lens f_i

$$\boxed{s_i=\frac{n_2}{n_2-n_1}R\equiv f_i.}$$

The focal length

Similarly, when

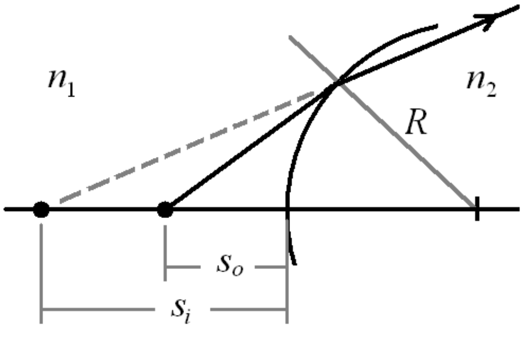
$$s_i \to \infty,$$

we may define the *focal length outside* the lens f_i

$$\boxed{s_o = \frac{n_1}{n_2 - n_1} R \equiv f_o.}$$

Special cases

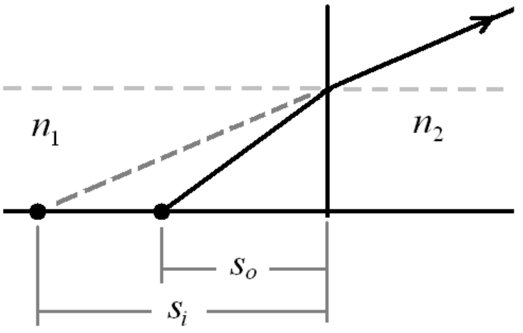
$$s_i < \infty.$$



In this case n_2/n_1 is not large enough to refract the transmitted ray below the horizontal and s_i marks a virtual image to the left of the lens.

Special cases

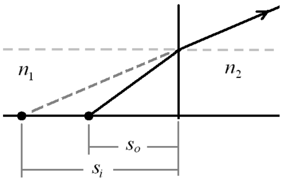
$$R \to \infty.$$



In this case, the curvature of the lens becomes zero, i.e. the lens surface becomes a flat plane

Special cases

$$R \to \infty.$$



We have

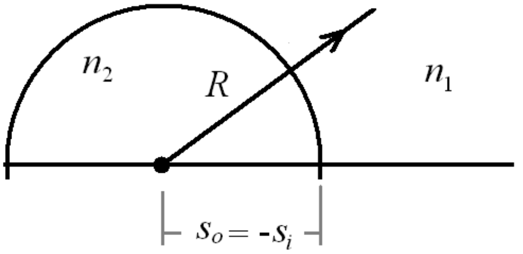
$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = 0,$$

giving

$$s_i = -\frac{n_2}{n_1} s_o.$$

Special cases

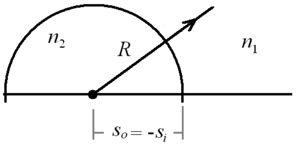
$$s_o = -s_i.$$



This can only be the case when the source of the ray is inside the lens.

Special cases

$$s_o = -s_i.$$

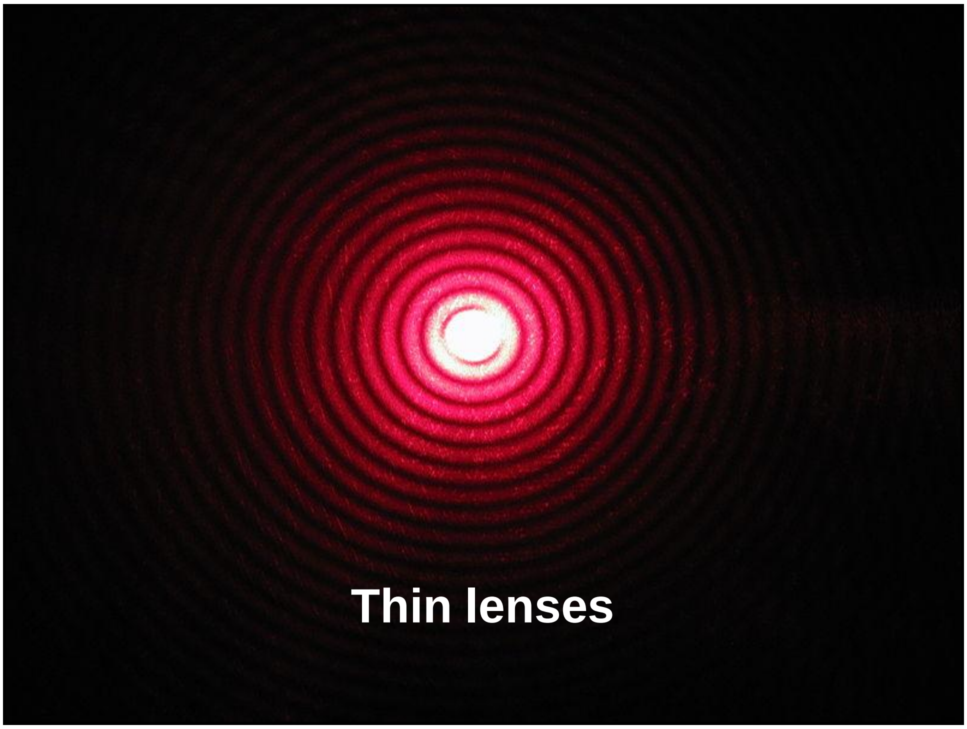


We have

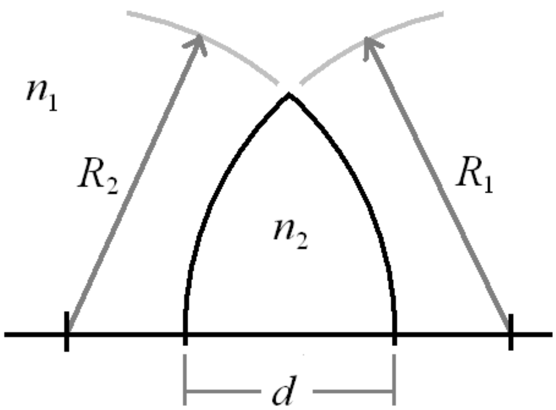
$$\frac{n_1}{s_o} - \frac{n_2}{s_o} = \frac{1}{R}(n_2 - n_1),$$

so

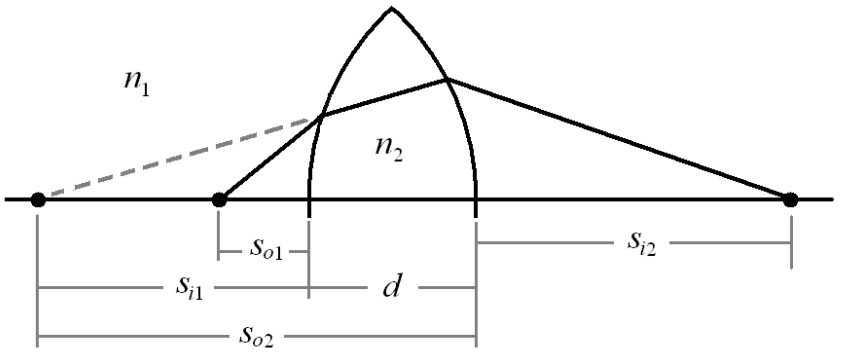
$$R = -s_o.$$



A thick lens



A thick lens



A thick lens

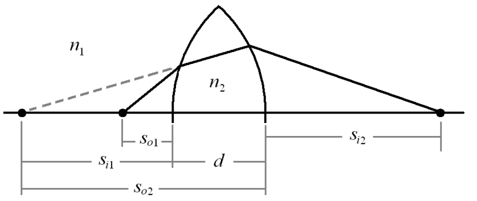
For the surface with radius of curvature R_1,

$$\frac{n_1}{s_{o1}} + \frac{n_2}{s_{i1}} = \frac{1}{R_1}(n_2 - n_1).$$

For the surface with radius of curvature R_2,

$$\frac{n_1}{s_{i2}} + \frac{n_2}{s_{o2}} = -\frac{1}{R_2}(n_2 - n_1).$$

A thick lens



Combining these results

$$n_1\left(\frac{1}{s_{i2}}+\frac{1}{s_{o1}}\right)=\left(\frac{1}{R_1}-\frac{1}{R_2}\right)(n_2-n_1)-n_2\left(\frac{1}{s_{i1}}+\frac{n_2}{s_{o2}}\right)$$

Inserting

$$s_{o2}=d-s_{i1},$$

The thin lens equation

$$n_1\left(\frac{1}{s_{i2}}+\frac{1}{s_{o1}}\right)=\left(\frac{1}{R_1}-\frac{1}{R_2}\right)(n_2-n_1)-\frac{n_2 d}{s_{i1}(d-s_{i1})}.$$

Taking the limit $d \to 0,$ and putting $s_{i2}=s_i, \quad s_{o2}=s_o,$

$$\boxed{\frac{1}{s_i}+\frac{1}{s_o}=\frac{n_2-n_1}{n_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right).}$$

This is the *thin lens equation*.

The lens maker's formula

If either $s_i \to \infty$ or $s_o \to \infty$

$$\frac{1}{s_o} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \text{ or } \frac{1}{s_i} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

But the term on the RHS is a constant, which we define to be the focal length f

$$\boxed{\frac{1}{f} = \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right).}$$

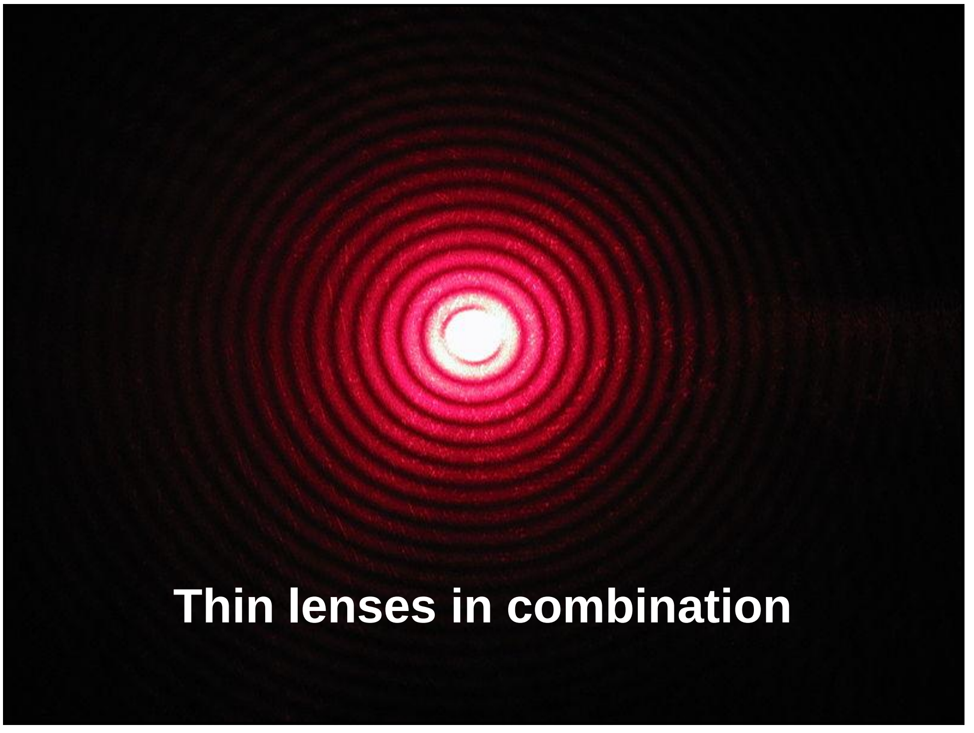
This is the *lens maker's formula*.

The Gaussian lens formula

We must also have

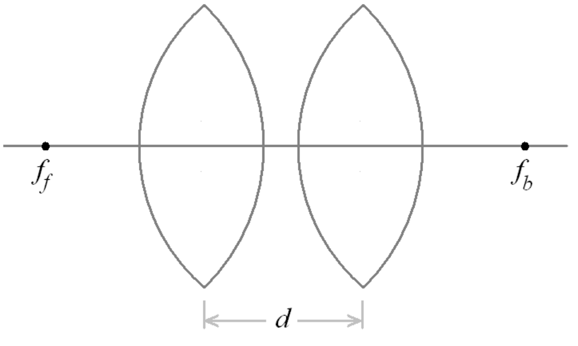
$$\boxed{\frac{1}{s_i} + \frac{1}{s_o} = \frac{1}{f}.}$$

This is the *Gaussian lens formula*.



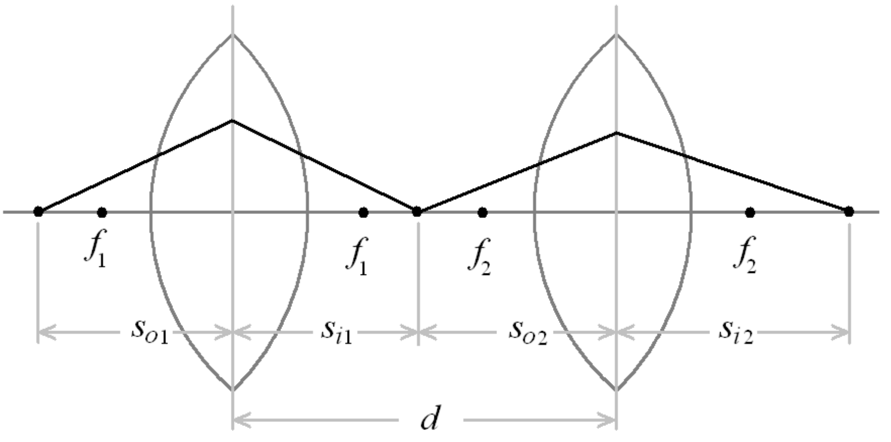
Thin lenses in combination

Consider two thin lenses separated by a distance d. We want expressions for the effective focal lengths of this combination.



We shall refer to these as the front and back focal lengths, f_f and f_b respectively.

Thin lenses in combination



Thin lenses in combination

The Gaussian lens formula for the first lens is

$$\frac{1}{s_{i1}} + \frac{1}{s_{o1}} = \frac{1}{f_1},$$

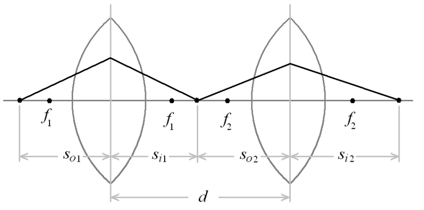
which gives

$$s_{i1} = \frac{s_{o1} f_1}{s_{o1} - f_1}.$$

Similarly, for the second lens

$$s_{i2} = \frac{s_{o2} f_2}{s_{o2} - f_2}.$$

Thin lenses in combination



Now

$$s_{o2} = d - s_{i1},$$

so

$$s_{i2} = \frac{(d - s_{i1}) f_2}{d - s_{i1} - f_2}.$$

Thin lenses in combination

Inserting

$$s_{i1} = \frac{s_{o1} f_1}{s_{o1} - f_1},$$

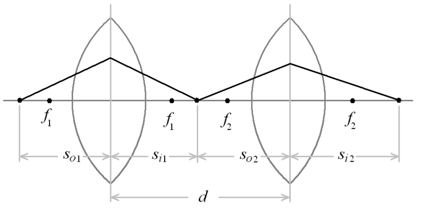
we obtain

$$s_{i2} = \frac{f_2 \left(d \left[s_{o1} - f_1 \right] - s_{o1} f_1 \right)}{s_{o1}(d - f_1 - f_2) - f_1 (d - f_2) d}.$$

Thin lenses in combination

If we take the limit

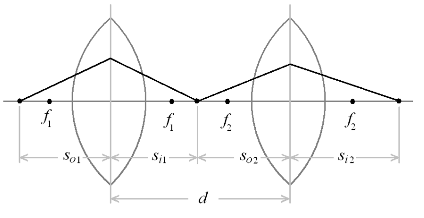
$$s_{o1} \to \infty,$$



s_{i2} becomes the back focal length and s_{i1} becomes f_1

$$f_b = \frac{(d - f_1) f_2}{d - f_1 - f_2}.$$

Thin lenses in combination



Similarly

$$f_f = \frac{(d - f_2) f_1}{d - f_1 - f_2}.$$

Thin lenses in close combination

If we let $d \to 0,$

$$f_b = \frac{f_1 f_2}{f_1 + f_2} = f_f .$$

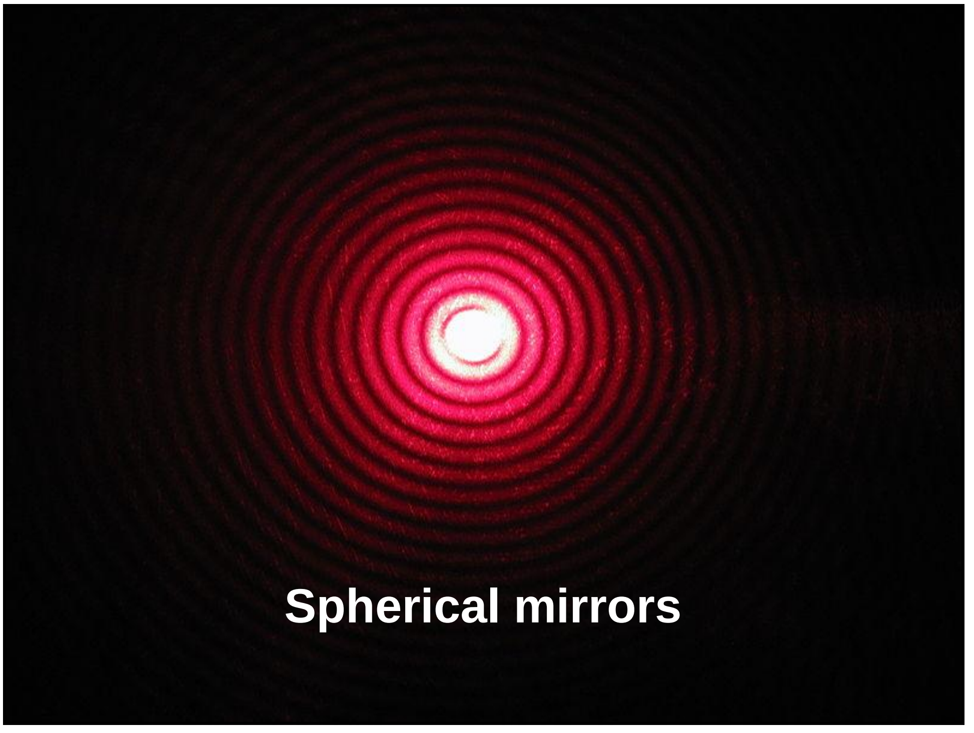
Putting $f_b = f$, this may be re-written as

$$\boxed{\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}.}$$

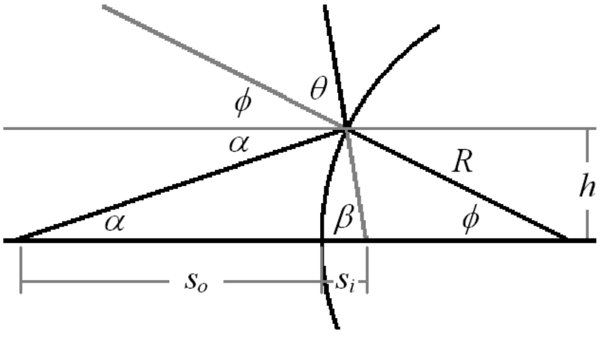
Thin lenses in close combination

This result may be generalised to a system of thin lenses in close combination

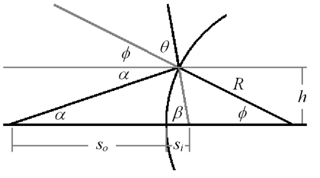
$$\boxed{\frac{1}{f} = \sum_i \frac{1}{f_i}.}$$



Mirror sign conventions

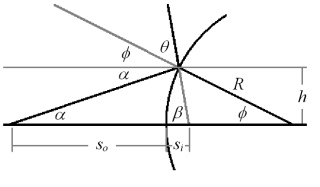


Mirror sign conventions



- Light is always taken to propagate from left to right.
- The object distance s_o is positive when it is to the left of the mirror surface.
- The image distance s_i is positive when it is to the left of the mirror surface (real image).
- The image distance s_i is negative when it is to the
- right of the mirror surface (virtual image).
- The radius R is positive if the mirror surface is to the right of the centre of the sphere (convex mirror)
- The radius R is negative if the mirror surface is to the left of the centre of the sphere (concave mirror)

Spherical mirrors



From the Law of Reflection

$$\theta = \alpha + \phi.$$

From inspection of the figure

$$\theta = \pi - \phi - (\pi - \beta) = \beta - \phi.$$

Hence

$$2\phi = \beta - \alpha.$$

Spherical mirrors

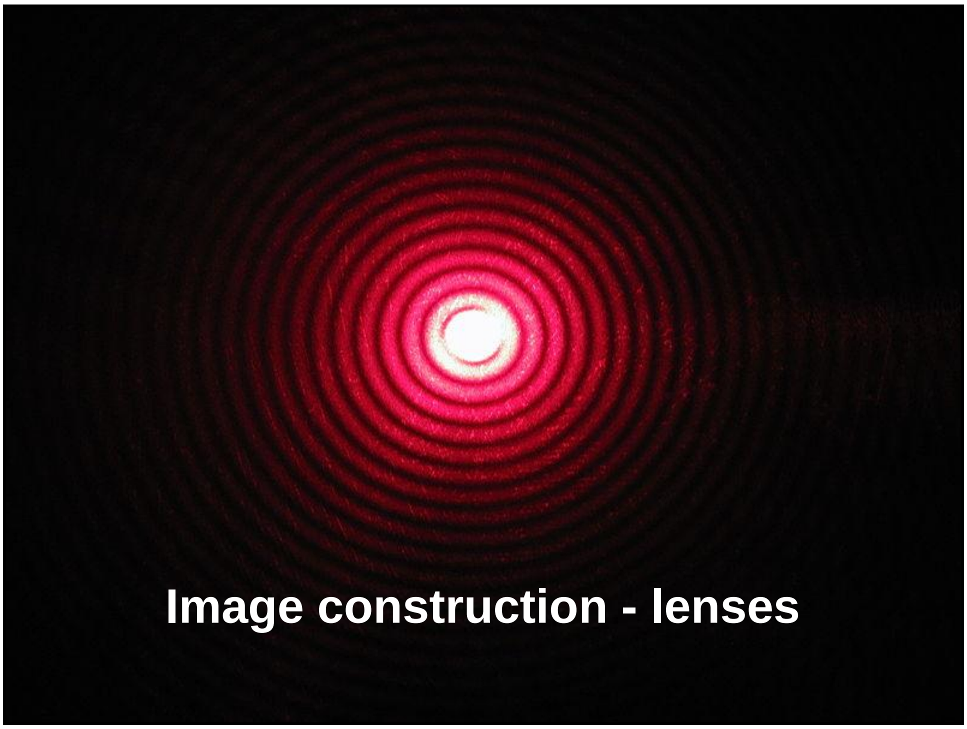
Hence

$$\boxed{\frac{1}{s_o}+\frac{1}{s_i}=-\frac{2}{R}.}$$

Taking limits as before and defining the focal length f, we then have

$$\boxed{\frac{1}{s_o}+\frac{1}{s_i}=\frac{1}{f},}$$

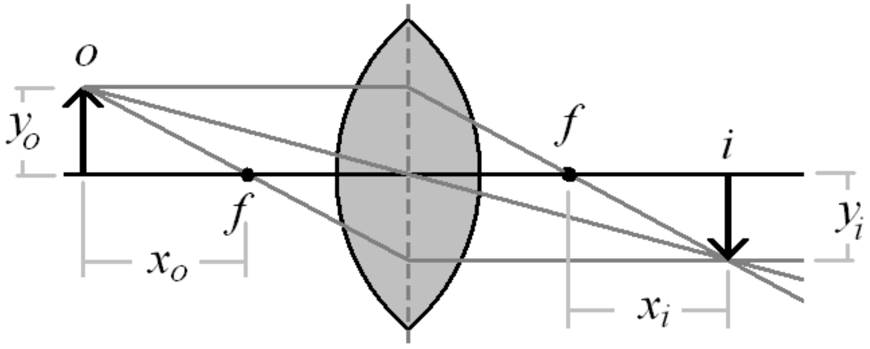
which is the same expression as the *Gaussian lens formula*.



Guide for convex lenses

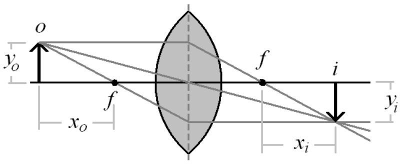
- Sketch a ray from the tip of the object parallel to the horizontal (the principle axis) to the centre line of the lens. From there, sketch another ray passing through the focus associated with the left-hand lens surface.
- Sketch a ray from the tip of the object directly through the centre of the lens without deviation.
- Sketch a ray from the tip of the object passing through the focus associated with the right-hand lens surface to the centre line of the lens. From there sketch a line parallel to the principle axis towards the image.

Convex lens



Convex lens

The image at i is **real** and **inverted**.



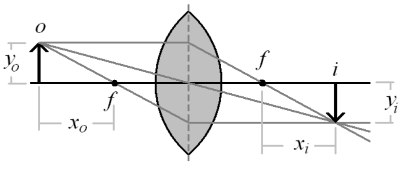
The magnification of the image M is given by

$$\boxed{M = \frac{y_i}{y_o}.}$$

Since y_i is negative, so is M (inverted image).

Convex lens

On the right-hand-side of the lens, we find similar triangles giving



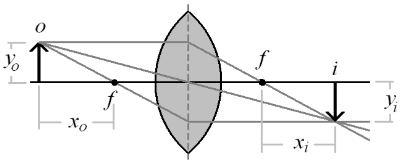
$$\frac{y_o}{f} = -\frac{y_i}{x_i}.$$

Hence

$$\boxed{M = -\frac{x_i}{f}.}$$

Convex lens

Similarly, we find



$$\boxed{M = -\frac{f}{x_o}.}$$

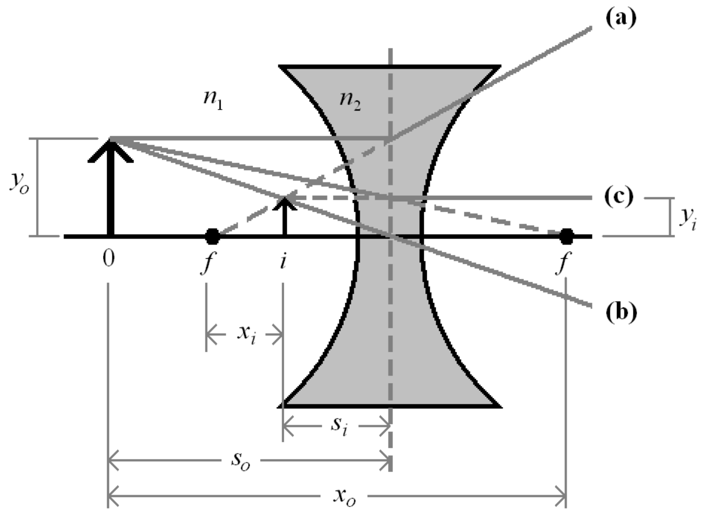
Thus

$$\boxed{f^2 = x_o x_i.}$$

Guide for concave lenses

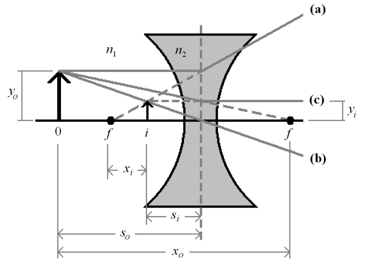
- **Sketch a ray parallel to the optical axis of the lens. Since $f < 0$, this must pass through f on the left of the lens (as a virtual ray).**
- **Sketch a ray passing through the centre of the lens without deviation**
- **Sketch a ray following the line through f on the right of the lens (this extension is virtual on the right) and emerging parallel to the optical axis. The parallel line is then extended to the left as a virtual ray**

Concave lens



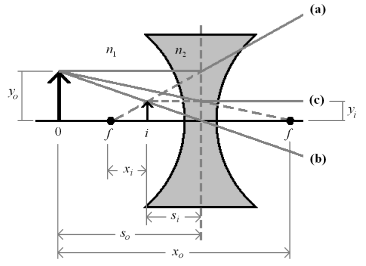
Concave lens

In this case, x_o is the distance between the object and f on the right of the lens.



x_i is the distance between the image and f on the left of the lens.

Concave lens

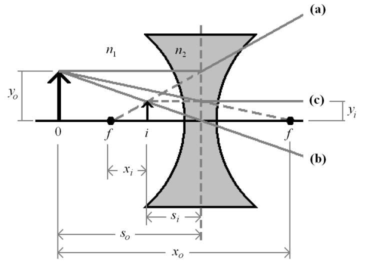


From similar triangles

$$\boxed{M = -\frac{x_i}{f}} \quad \text{and} \quad \boxed{M = -\frac{f}{x_o},}$$

as we had for the convex lens.

Concave lens



Thus we obtain the same result

$$\boxed{f^2 = x_o x_i .}$$