PY4118 Physics of Semiconductor Devices

Kronig-Penney Solution

When evaluating Kronig-Penney, we end up with four linear equations.

The 4x4 linear equation to be solved is: $\begin{bmatrix} 1 & 1 & -1 & -1 \\ i\beta & -i\beta & -\alpha & \alpha \\ e^{i\beta w} & e^{-i\beta w} & -e^{-\alpha t}e^{ika} & -e^{\alpha t}e^{ika} \\ i\beta e^{i\beta w} & -i\beta e^{-i\beta w} & -\alpha e^{-\alpha t}e^{ika} & \alpha e^{\alpha t}e^{ika} \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$

Thus we need to solve: $\det\begin{bmatrix} 1 & 1 & -1 & -1 \\ i\beta & -i\beta & -\alpha & \alpha \\ e^{i\beta w} & e^{-i\beta w} & -e^{-\alpha t}e^{ika} & -e^{\alpha t}e^{ika} \\ i\beta e^{i\beta w} & -i\beta e^{-i\beta w} & -\alpha e^{-\alpha t}e^{ika} & \alpha e^{\alpha t}e^{ika} \end{bmatrix} = 0$

This will be broken into 4 pieces:

1) $\begin{bmatrix} \boxed{1} & 1 & -1 & -1 \\ i\beta & -i\beta & -\alpha & \alpha \\ e^{i\beta w} & e^{-i\beta w} & -e^{-\alpha t}e^{ika} & -e^{\alpha t}e^{ika} \\ i\beta e^{i\beta w} & -i\beta e^{-i\beta w} & -\alpha e^{-\alpha t}e^{ika} & \alpha e^{\alpha t}e^{ika} \end{bmatrix}$

$$\begin{aligned}
&1\times\left[2i\beta\alpha e^{2ika} + \alpha\left(\alpha e^{\alpha t}e^{ika}e^{-i\beta w} - i\beta e^{\alpha t}e^{ika}e^{-i\beta w}\right) + \alpha\left(-\alpha e^{-\alpha t}e^{ika}e^{-i\beta w} - i\beta e^{-\alpha t}e^{ika}e^{-i\beta w}\right)\right] \\
&= 2i\beta\alpha e^{2ika} + \alpha^2\left(e^{\alpha t}e^{ika}e^{-i\beta w} - e^{-\alpha t}e^{ika}e^{-i\beta w}\right) - i\beta\alpha\left(e^{\alpha t}e^{ika}e^{-i\beta w} + e^{-\alpha t}e^{ika}e^{-i\beta w}\right) \\
&= 2i\beta\alpha e^{2ika} + 2\alpha^2 e^{ika}e^{-i\beta w}\sinh(\alpha t) - 2i\beta\alpha e^{ika}e^{-i\beta w}\cosh(\alpha t)
\end{aligned}$$

2) $\begin{bmatrix} 1 & \boxed{1} & -1 & -1 \\ i\beta & -i\beta & -\alpha & \alpha \\ e^{i\beta w} & e^{-i\beta w} & -e^{-\alpha t}e^{ika} & -e^{\alpha t}e^{ika} \\ i\beta e^{i\beta w} & -i\beta e^{-i\beta w} & -\alpha e^{-\alpha t}e^{ika} & \alpha e^{\alpha t}e^{ika} \end{bmatrix}$

$$\begin{aligned}
&-1\times\left[i\beta\left(-2e^{2ika}\right) + \alpha\left(\alpha e^{\alpha t}e^{ika}e^{i\beta w} + i\beta e^{i\beta w}e^{\alpha t}e^{ika}\right) + \alpha\left(-\alpha e^{-\alpha t}e^{ika}e^{i\beta w} + i\beta e^{-\alpha t}e^{ika}e^{i\beta w}\right)\right] \\
&= 2i\beta e^{2ika} - \alpha^2\left(e^{\alpha t}e^{ika}e^{i\beta w} - e^{-\alpha t}e^{ika}e^{i\beta w}\right) - i\beta\alpha\left(e^{i\beta w}e^{\alpha t}e^{ika} + e^{-\alpha t}e^{ika}e^{i\beta w}\right) \\
&= 2i\beta e^{2ika} - 2\alpha^2 e^{ika}e^{i\beta w}\sinh(\alpha t) - 2i\beta\alpha e^{ika}e^{i\beta w}\cosh(\alpha t)
\end{aligned}$$

Adding equations 1) + 2) yields:

$$4i\beta\alpha e^{2ika} - 4i\alpha^2 e^{ika}\sinh(\alpha t)\sin(\beta w) - 4i\beta\alpha e^{ika}\cosh(\alpha t)\cos(\beta w) \qquad \textbf{(A)}$$

$$
3)\quad \begin{bmatrix} 1 & 1 & \boxed{-1} & -1 \\ i\beta & -i\beta & -\alpha & \alpha \\ e^{i\beta w} & e^{-i\beta w} & -e^{-\alpha t}e^{ika} & -e^{\alpha t}e^{ika} \\ i\beta e^{i\beta w} & -i\beta e^{-i\beta w} & -\alpha e^{-\alpha t}e^{ika} & \alpha e^{\alpha t}e^{ika} \end{bmatrix}
$$

$$
\begin{aligned}
&-1\times\left[i\beta\left(\alpha e^{\alpha t}e^{ika}e^{-i\beta w}-i\beta e^{\alpha t}e^{ika}e^{-i\beta w}\right)+i\beta\left(\alpha e^{\alpha t}e^{ika}e^{i\beta w}+i\beta e^{\alpha t}e^{ika}e^{i\beta w}\right)-i\beta\alpha(1+1)\right]\\
&=2i\beta\alpha-i\beta\alpha\left(e^{\alpha t}e^{ika}e^{i\beta w}+e^{\alpha t}e^{ika}e^{-i\beta w}\right)-(i\beta)^2\left(e^{\alpha t}e^{ika}e^{i\beta w}-e^{\alpha t}e^{ika}e^{-i\beta w}\right)\\
&=2i\beta\alpha-2i\beta\alpha e^{\alpha t}e^{ika}\cos(\beta w)-2i\beta^2e^{\alpha t}e^{ika}\sin(\beta w)
\end{aligned}
$$

$$
4)\quad \begin{bmatrix} 1 & 1 & -1 & \boxed{-1} \\ i\beta & -i\beta & -\alpha & \alpha \\ e^{i\beta w} & e^{-i\beta w} & -e^{-\alpha t}e^{ika} & -e^{\alpha t}e^{ika} \\ i\beta e^{i\beta w} & -i\beta e^{-i\beta w} & -\alpha e^{-\alpha t}e^{ika} & \alpha e^{\alpha t}e^{ika} \end{bmatrix}
$$

$$
\begin{aligned}
&1\times\left[i\beta\left(-\alpha e^{-\alpha t}e^{ika}e^{-i\beta w}-i\beta e^{-\alpha t}e^{ika}e^{-i\beta w}\right)+i\beta\left(-\alpha e^{-\alpha t}e^{ika}e^{i\beta w}+i\beta e^{-\alpha t}e^{ika}e^{i\beta w}\right)+i\beta\alpha(1+1)\right]\\
&=2i\beta\alpha-i\beta\alpha\left(e^{-\alpha t}e^{ika}e^{i\beta w}+e^{-\alpha t}e^{ika}e^{-i\beta w}\right)+(i\beta)^2\left(e^{-\alpha t}e^{ika}e^{i\beta w}-e^{-\alpha t}e^{ika}e^{-i\beta w}\right)\\
&=2i\beta\alpha-2i\beta\alpha e^{-\alpha t}e^{ika}\cos(\beta w)+2i\beta^2e^{-\alpha t}e^{ika}\sin(\beta w)
\end{aligned}
$$

Adding equations 3) + 4) yields:

$$
4i\beta\alpha-4i\beta\alpha e^{ika}\cosh(\alpha t)\cos(\beta w)-4i\beta^2e^{ika}\sinh(\alpha t)\sin(\beta w) \qquad \textbf{(B)}
$$

Now adding equations (A) and (B)

The determinate now reduced to:

$$
\begin{aligned}
&4i\beta\alpha e^{2ika}-4i\alpha^2e^{ika}\sinh(\alpha t)\sin(\beta w)-4i\beta\alpha e^{ika}\cosh(\alpha t)\cos(\beta w)\\
&+4i\beta\alpha-4i\beta\alpha e^{ika}\cosh(\alpha t)\cos(\beta w)-4i\beta^2e^{ika}\sinh(\alpha t)\sin(\beta w)=0
\end{aligned}
$$

Simplifying…

$$
\begin{aligned}
&\beta\alpha\left(e^{ika}+e^{-ika}\right)-\left(\alpha^2+\beta^2\right)\sinh(\alpha t)\sin(\beta w)-2\beta\alpha\cosh(\alpha t)\cos(\beta w)=0\\
&2\beta\alpha\cos(ka)=\left(\alpha^2+\beta^2\right)\sinh(\alpha t)\sin(\beta w)+2\beta\alpha\cosh(\alpha t)\cos(\beta w)
\end{aligned}
$$

And finally:

$$
\cos(ka)=\frac{\alpha^2+\beta^2}{2\beta\alpha}\sinh(\alpha t)\sin(\beta w)+\cosh(\alpha t)\cos(\beta w)
$$

And, this is the published result.